Steady mirror structures in a plasma with pressure anisotropy

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(Dated:)

Abstract

In the first part of this paper we present a review of our results concerning the weakly nonlinear regime of the mirror instability in the framework of an asymptotic model. This model belongs to the class of gradient type systems for which the free energy can only decrease in time. It reveals a behavior typical for subcritical bifurcations: below the mirror instability threshold, all localized stationary structures are unstable, while above threshold, the system displays a blow-up behavior. It is shown that taking the electrons into account (non-zero temperature) does not change the structure of the asymptotic model. For bi-Maxwellian distributions functions for both electrons and ions, the model predicts the formation of magnetic holes. The second part of this paper contains original results concerning two-dimensional steady mirror structures which can form in the saturated regime. In particular, based on Grad-Shafranov-like equations, a gyrotropic plasma, where the pressures in the static regime are only functions of the amplitude of the local magnetic field, is shown to be amenable to a variational principle with a free energy density given by the parallel tension. This approach is used to demonstrate that small-amplitude static holes constructed slightly below the mirror instability threshold identify with lump solitons of KPII equation and turn out to be unstable. It is also shown that regularizing effects such as finite Larmor radius corrections cannot be ignored in the description of large-amplitude mirror structures. Using the gradient method, which is based on a variational principle for anisotropic MHD taking into account ion finite Larmor radius effects, we found both one-dimensional magnetic structures in the form of stripes and two-dimensional bubbles when the magnetic field component transverse to the plane is increased. These structures realize minimum of the free energy.

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PACS: 52.35.Py, 52.25.Xz, 94.30.cj, 94.05.-a

I. INTRODUCTION

Magnetic structures in the form of holes or humps associated with maxima or minima of plasma density and pressure are often encountered in planetary magnetosheafs close to both the bow-shock and the magnetopause, and in the solar wind (see e.g. [1–3]) as well. These structures are often viewed as ultra-low frequency (ULF) waves resulting from the mirror instability (MI) [4], and, by this reason, called mirror structures. This instability develops in a collisionless plasma characterized by a relatively large β (a few units) and a transverse (usually ionic) temperature $T_\perp$ larger than the parallel one $T_\|$, such that the condition for mirror instability

$$T_\perp/T_\| - 1 > \beta_\perp^{-1} \tag{1}$$

is fulfilled. Here $\beta_\perp = 8\pi p_\perp/B^2$ (similarly, $\beta_\| = 8\pi p_\|/B^2$), where $p_\perp = nT_\perp$and $p_\| = nT_\|$ are the perpendicular and parallel plasma pressures respectively.

In the Earth magnetosheath, a typical depth of magnetic holes is about 20% of the mean magnetic field value and can sometimes achieve 50 %. The characteristic width of such structures is of the order of a few ion Larmor radii, and they display an aspect ratio of about 7-10. In solar wind, according to [3], the size of holes may be very different, varying from 10 up to 1000 ion gyroradii. In magnetosheath, holes and humps have comparable size. and amplitudes. Humps are often observed near the magnetopause where conditions (1) for development of the MI can be met under the effect of the plasma compression. Mirror structures are also observed when the plasma is linearly stable [6, 7], which may be viewed as the signature of a bistability regime resulting from a subcritical bifurcation, whose existence was interpreted on the basis of a simple energetic argument within the simplified description of anisotropic magnetohydrodynamics [8].

The linear mirror instability has been extensively studied both analytically (see e.g. [9, 10]), and by means of particle-in-cell (PIC) simulations [11]. As shown in [9, 12, 13], the instability is arrested at large k due to finite ion Larmor radius (FLR) effects. It turns out that wave-particle resonance plays a central role in driving the instability, while the FLR effects are at the origin of the quenching of the instability at small scales. In contrast, a few years ago, a theoretical understanding of the nonlinear phase remained limited to phenomenological modeling of particle trapping [14, 15] that hardly reproduce simulations of Vlasov-Maxwell equations [16].

The first nonlinear theory was formulated in [17, 18] where we developed a weakly nonlinear approach to the mirror instability based on the mixed hydrodynamic-kinetic description. For the sake of simplicity, an electron-proton plasma with cold electrons was considered first. It includes the force-balance equation within the anisotropic MHD and the drift kinetic equation for the ions. Close to threshold, the unstable modes have wavevectors almost perpendicular to the ambient magnetic field $\mathbf{B}$ ($k_z/k_\perp \ll 1$) with $k_\perp \rho_i \ll 1$ (ρ_i is the ion Larmor radius), so that the perturbations can be described using a long-wave approximation. The latter allows one to apply the drift kinetic equation (see, e.g. [19, 20]) to estimate the main nonlinear effects that correspond to a local shift of the instability threshold (1). All other nonlinearities connected, for example, with ion inertia are smaller. As the result, it is possible to derive an asymptotic equation with quadratic nonlinearity of generalized gradient type [17, 18]. The latter property implies an irreversible character of the mirror modes behavior, associated with ion Landau damping, where the free energy (or Lyapunov functional) can only decrease in time. In this framework, above threshold, the mirror modes have a blow-up behavior with a possible saturation at an amplitude level comparable to that of the ambient field. Below threshold, all stationary (localized) structures were predicted to be unstable. Thus, the system near the MI threshold displays a behavior typical of a subcritical bifurcation when the small-amplitude stationary solutions below threshold turn out to be unstable; above threshold, the amplitude of magnetic field perturbations tends to blow up. It is worth noting that this approach contrasts with the quasi-linear theory [22] that also assumes vicinity of the instability threshold but, being based on a random phase approximation, cannot predict the appearance of coherent structures. Phenomenological models based on the cooling of trapped particles were proposed to interpret the existence of deep magnetic holes [14, 23]These models do not however address the initial value problem in the mirror unstable regime.

The asymptotic model [17, 18] was first derived under the assumption of cold electrons. Therefore, in our further papers [24, 25], we considered how hot electrons can be incorporated into the model. The approach we developed is based on the assumption of an adiabatically slow dynamics of the mirror structures that allows one to compute the coupling coefficient in the weakly nonlinear regime as well as to simplify all calculations of the linear growth rate in the case of bi-Maxwellian distributions for both the ions and the electrons. The adiabatic hypothesis can be proved perturbatively, and is in particular valid within the asymptotic

model. Because this model predicts the existence of subcritcal bifurcation with a blow-up behavior above threshold, consistent with the formation of mirror structures with amplitude of the magnetic field perturbation comparable with the ambient field, our next step was to investigate the properties of possible stationary mirror structures.

The aim of the present paper is twofolds. The first part provides a review of our previous results concerning the weakly nonlinear model for both cold ([17, 18]) and hot ([24]) electrons. Another goal of this paper is to study steady mirror structures resulting from the balance of magnetic and (both parallel and perpendicular) thermal pressures, whose simplest description is provided by anisotropic MHD. Isotropic MHD equilibria are classically governed by the Grad-Shafranov (GS) equation [26, 27, 29]. We here revisit this approach in the case of anisotropic electron and ion fluids where the perpendicular and parallel pressures are given by equations of state appropriate for the static character of the solutions. However, the MHD stationary equations, at least in the two-dimensional geometry, turn out to be ill-posed. As a consequence, these equations require some regularization. As done in a similar context of pattern formation [30], an additional linear term involving a square Laplacian is added. For nonlinear mirror modes, regularization can originate from finite Larmor radius (FLR) corrections, which are not retained in the present analysis based on the drift kinetic equation (see, e.g. [17, 18]).

The paper is organized as follows. In Section II, we discuss the linear mirror instability near the MI threshold. Section III is devoted to the derivation of weakly nonlinear asymptotic model, in the simplest case of cold electrons, and to its properties, including possible stationary states (below the MI threshold) and blow-up behavior (above threshold). Section IV deals with accounting electrons in the asymptotic model. Here we develop the adiabatic approach for finding contributions from electrons to both the linear growth rate and the nonlinear coupling coefficient entering the asymptotic model. In Section V, we formulate the variational principle for the stationary anisotropic MHD when both parallel and transverse pressures depend on the magnetic field amplitude with a free energy given by the space integral of the parallel tension. In this case, as well known [31–36], the parallel component of the MHD equation is satisfied identically. In Section 6, the anisotropic Grad-Shafranov equations are revisited when the gyrotropic pressures depend only on the local magnetic field amplitude that, as shown in the forthcoming sections, is specific of nonlinear mirror modes. In this case the stationary anisotropic MHD represents an hydrodynamic integrable-type

system and for this reason requires the renormalization due to FLR effects. In this Section, it is shown also that the equations of state resulting from an adiabatic approximation of the drift kinetic description, require a regularization because of an overestimate of the contributions from the particles with a large magnetic moment. We discuss in particular the small-amplitude regime and show that the pressure-balanced structures are then governed by the KPII equation which possesses lump solutions. Numerical simulations reproduce these special structures, that turn out to be unstable. Computation of stable solutions lead to large-amplitude purely one-dimensional solutions in the form of stripes that appear to be sensitive to the regularization process, an indication that the regime cannot be captured by the drift kinetic approximation and that finite Larmor corrections and trapped particles are to be retained. Section VII aims for presentation of the numerical results for two-dimensional (depending on x and y coordinates) stationary mirror structures when the magnetic field $\mathbf{B}$ has also a B_z component. In particular, we show that for small B_z stationary structures realizing the minimum of the free energy, below and above the threshold, have the form of stripes which are one-dimensional structures with constant magnetic field outside and inside the stripes. The transient region, between outer and inner regions, for the stripes represents the magnetic well which structure is defined by the FLR contributions to the free energy. With increasing B_z, instead of stripes, the free energy has its minimum for bubble-type structures with an elliptic form. When $B_{x,y} \to 0$ these bubbles become circular. In this case, FLR effects play a role of the surface tension. Section VIII is the conclusion.

II. MAIN EQUATIONS AND MIRROR INSTABILITY

Consider for the sake of simplicity, a plasma with cold electrons. To describe the mirror instability in the long-wave limit it is enough to use the drift kinetic equation for ions ignoring parallel electric field $E_\parallel$ and transverse electric drift:

$$\frac{\partial f}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla f - \mu \mathbf{b} \cdot \nabla B \frac{\partial f}{\partial v_\parallel} = 0. \tag{2}$$

In this approximation ions move along the magnetic field ($\mathbf{b} = \mathbf{B}/B$) due to the magnetic force $\mu\, \mathbf{b} \cdot \nabla B$ where $\mu = v_\perp^2/(2B)$ is the adiabatic invariant which plays the role of a parameter in equation (2). Both pressures $p_\parallel$ and $p_\perp$ are given by

$$p_\parallel = mB \int v_\parallel^2 f d\mu dv_\parallel d\varphi \equiv m \int v_\parallel^2 f d^3 v, \tag{3}$$

$$p_\perp = mB^2 \int \mu f d\mu dv_\| d\varphi \equiv \frac{1}{2} m \int v_\perp^2 f d^3 v. \tag{4}$$

Equation (2) with relations (3) and (4) are supplemented with the equation expressing the balance of forces in a plane transverse to the local magnetic field

$$\begin{aligned} &\hat{\Pi}\Big\{ -\nabla\Big(p_\perp + \frac{B^2}{8\pi}\Big) \\ &+\Big[1 + \frac{4\pi}{B^2}\Big(p_\perp - p_\|\Big)\Big]\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}\Big\} = 0. \end{aligned} \tag{5}$$

Here, consistently with the long-wave approximation, we neglect both the plasma inertia and the non-gyrotropic contributions to the pressure tensor. Furthermore, $\hat{\Pi}_{ik} = \delta_{ik} - b_i b_k$ denotes the projection operator in the plane transverse to the local magnetic field. In this equation, the first term describes the action of the magnetic and perpendicular pressures, the second term being responsible for magnetic lines elasticity.

The equation governing the mirror dynamics is then obtained perturbatively by expanding Eqs. (2) and (5). In this approach, the ion pressure tensor elements are computed from the system (2), (5), near a bi-Maxwellian equilibrium state characterized by temperatures $T_\perp$ and $T_\|$ and a constant ambient magnetic field $\mathbf{B_0}$ taken along the z-direction.

From Eq. (5) linearized about the background field $\mathbf{B_0}$ by writing $\mathbf{B} = \mathbf{B}_0 + \widetilde{\mathbf{B}}$ ($B_0 \gg \widetilde{B}$) with $\widetilde{\mathbf{B}} \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}$, we have

$$p_\perp^{(1)} + \frac{B_0 \widetilde{B}_z}{4\pi} = -\frac{k_z^2}{k_\perp^2}\left(1 + \frac{\beta_\perp - \beta_\|}{2}\right)\frac{B_0 \widetilde{B}_z}{4\pi}. \tag{6}$$

Here k_z and $k_\perp$ are the projections of the wave vector $\mathbf{k}$, and $p_\perp^{(1)}$ is calculated from the linearized drift kinetic equation (2):

$$\frac{\partial f^{(1)}}{\partial t} + v_\| \frac{\partial f^{(1)}}{\partial z} - \mu \frac{\partial \widetilde{B}_z}{\partial z}\frac{\partial f^{(0)}}{\partial v_\|} = 0.$$

In Fourier space, this equation has the solution

$$f^{(1)} = -\frac{\mu \widetilde{B}_z}{\omega - k_z v_\|} k_z \frac{\partial f^{(0)}}{\partial v_\|}. \tag{7}$$

The mirror instability is such that $\omega/k_z \ll v_{th\|} = \sqrt{2T_\|/m}$. This means that the ions contributing to the resonance $\omega - k_z v_\| = 0$, correspond to the maximum of the ion distribution function.

After substituting (7) into the first order term for perpendicular pressure (4) and performing integration, we get

$$p_\perp^{(1)} = \beta_\perp \left(1 - \frac{\beta_\perp}{\beta_\|}\right) \frac{B_0 \widetilde{B}_z}{4\pi} - \frac{i\sqrt{\pi}\omega}{|k_z| v_{th\|}} \frac{\beta_\perp^2}{\beta_\|} \frac{B_0 \widetilde{B}_z}{4\pi}. \tag{8}$$

The first term in (8) is due to the difference between perpendicular and parallel pressures, while the second one accounts for the Landau pole.

Equation (8) together with (6) yield the growth rate for the mirror instability in the drift approximation where FLR corrections are neglected [4]

$$\gamma = |k_z| v_{th\|} \frac{\beta_\|}{\sqrt{\pi}\beta_\perp} \left[\frac{\beta_\perp}{\beta_\|} - 1 - \frac{1}{\beta_\perp} - \frac{k_z^2}{k_\perp^2 \beta_\perp}\chi\right], \tag{9}$$

where $\chi = 1 + (\beta_\perp - \beta_\|)/2$. The instability takes place when the criterion (1) is fulfilled and, near threshold, develops in quasi-perpendicular directions, making the parallel magnetic perturbation dominant.

As shown in Refs. [9, 12, 13], when the FLR corrections are relevant, the growth rate is modified into

$$\gamma = |k_z| v_{th\|} \frac{\beta_\| \chi}{\sqrt{\pi}\beta_\perp^2} \left[\varepsilon - \frac{k_z^2}{k_\perp^2} - \frac{3}{4\chi} k_\perp^2 \rho_i^2\right] \tag{10}$$

where $\varepsilon = \beta_\perp \chi^{-1}(\beta_\perp/\beta_\| - 1 - \beta_\perp^{-1})$ and the ion Larmor radius $\rho_i = v_{th\perp}/\omega_{ci}$ is defined with the transverse thermal velocity $v_{th\perp} = \sqrt{2T_\perp/m}$ and the ion gyrofrequency $\omega_{ci} = eB_0/(mc)$. This growth rate can be recovered by expanding the general expression given in [9], in the limit of small transverse wavenumbers. It can also be obtained directly from the Vlasov-Maxwell (VM) equations in a long-wave limit which retains non gyrotropic contributions [37]. It is important to note that the expression (10) for γ is consistent with the applicability condition $\omega/k_z \ll v_{th\|}$, i.e. when the supercritical parameter $|\varepsilon| \ll 1$. In this case the instability saturation happens at small $k_\perp \propto \sqrt{\varepsilon}$ due to FLR and for almost perpendicular direction in a small cone of angles, $k_z/\ k_\perp \propto \sqrt{\varepsilon}$. As a result, the growth rate $\gamma \propto \varepsilon^2$, so that, when defining new stretched variables by

$$\begin{aligned} k_z &= \varepsilon K_z \rho_i^{-1} (2/\sqrt{3}) \chi^{1/2}, \\ k_\perp &= \sqrt{\varepsilon}(2/\sqrt{3}) K_\perp \rho_i^{-1} \chi^{1/2}, \\ \gamma &= \varepsilon^2 \Gamma (2/\sqrt{3}) \Omega \left(\sqrt{\pi}\beta_\perp\right)^{-1} \left(\chi \beta_\| / \beta_\perp\right)^{3/2}, \end{aligned} \tag{11}$$

it takes the form

$$\Gamma = |K_z| \left(1 - K_z^2/K_\perp^2 - K_\perp^2\right). \tag{12}$$

Hence it is seen that, in the $(K_\perp - \Theta)$ plane $(\Theta \equiv K_z/K_\perp)$, the instability takes place inside the unit circle: $\Theta^2 + K_\perp^2 < 1$. The maximum of Γ is obtained for $K_\perp = 1/2$, $\Theta = \pm\, 1/2$ and is equal to $\Gamma_{\max} = 1/8$. Outside the circle, the growth rate becomes negative (in agreement with [13]).

III. WEAKLY NONLINEAR REGIME: ASYMPTOTIC MODEL FOR COLD ELECTRONS

A. Derivation

As it follows from (6), in the linear regime, near the instability threshold, the fluctuations of perpendicular and magnetic pressures almost compensate each other (compare with (9)). Therefore, in the nonlinear stage of this instability, we can expect that the main nonlinear contributions come from the second order corrections to the total (perpendicular plus magnetic) pressure, i.e.

$$p_\perp^{(1)} + \frac{B_0 \widetilde{B}_z}{4\pi} + p_\perp^{(2)} + \frac{\widetilde{B}_z^2}{8\pi} = -\chi \frac{\partial_z^2}{\Delta_\perp} \frac{B_0 \widetilde{B}_z}{4\pi}. \tag{13}$$

This result can be obtained rigorously by means of a multi-scale expansion based on the linear theory scalings (11). For this purpose, we introduce a slow time T and slow coordinates $\mathbf{R}$ in a way consistent with (11), and expand the magnetic field fluctuations as a powers series in $\varepsilon^{1/2}$:

$$\widetilde{B}_z = \varepsilon B_z^{(1)} + O(\varepsilon^2), \;\; \widetilde{\mathbf{B}}_\perp = \varepsilon^{3/2} \mathbf{B}_\perp^{(3/2)} + O(\varepsilon^{5/2}), \tag{14}$$

where $\mathbf{B}^{(n/2)}$ are assumed to be functions of $\mathbf{R}$ and T. Using these expressions, it is easy to establish that quadratic nonlinear terms coming from the expansion of Π in (5) as well as from the second term in the r.h.s. of Eq. (13) are small in comparison with the quadratic term originating from the magnetic pressure in Eq. (13). Thus, to get a nonlinear model for mirror dynamics, it is enough to find $p_\perp^{(2)}$. The expansion (14) induces a corresponding expansion for the distribution function and for both pressures. Defining

$$\widetilde{p}_\perp^{(n)} = \pi m \int v_\perp^2 f^{(n)} v_\perp dv_\perp dv_\|,$$

from (4) we have

$$p_\perp^{(2)} = (B_z^{(1)}/B_0)^2 p_\perp^{(0)} + 2(B_z^{(1)}/B_0)\, \widetilde{p}_\perp^{(1)} + \widetilde{p}_\perp^{(2)},$$

up to an additional contribution proportional to $B_z^{(2)}$ that cancels out in the final equation due to the threshold condition.

On the considered time scale, the effect of nonlinear Landau resonance is negligible in the contribution to $f^{(2)}$ that can thus be estimated from the equation

$$v_\parallel \frac{\partial f^{(2)}}{\partial z} + (2\mu^2/v_{th\parallel}^2) B_z^{(1)} \frac{\partial B_z^{(1)}}{\partial z} \frac{\partial f^{(0)}}{\partial v_\parallel} = 0.$$

For an equilibrium bi-Maxwellian distribution, we have

$$f^{(2)} = (2\mu^2/v_{th\parallel}^4)(B_z^{(1)})^2 f^{(0)} \tag{15}$$

and thus

$$p_\perp^{(2)} = \left(\beta_\perp - 4\beta_\perp^2/\beta_\parallel + 3\beta_\perp^3/\beta_\parallel^2\right) \frac{\widetilde{B}_z^2}{8\pi}.$$

As a consequence, because of the vicinity to threshold we obtain

$$p_\perp^{(2)} + \frac{\widetilde{B}_z^2}{8\pi} = \left(1 + \beta_\perp^{-1}\right) \frac{3\widetilde{B}_z^2}{8\pi} > 0. \tag{16}$$

Then rewriting equation (13) using the slow variables (11) and rescaling the amplitude

$$\widetilde{B}_z/B_0 = \varepsilon 2\chi\beta_\perp (1+\beta_\perp)^{-1} u,$$

we arrive at the equation [17, 18]

$$\frac{\partial u}{\partial T} = \widehat{K}_Z \left[\left(\sigma - \Delta_\perp^{-1} \frac{\partial^2}{\partial Z^2} + \Delta_\perp\right) u - 3u^2\right]. \tag{17}$$

Here $\sigma = \pm 1$, depending of the positive or negative sign of ε, $\widehat{K}_Z = -\mathcal{H}\partial_Z$ is a positive definite operator (whose Fourier transform is $|K_Z|$), $\widehat{H}$ is Hilbert transform:

$$\widehat{H} f(Z) = \frac{1}{\pi} VP \int_{-\infty}^{\infty} \frac{f(Z')}{Z' - Z} dZ'.$$

As seen from the equation, its linear part reproduces the growth rate (12). In particular, the third term in the r.h.s. accounts for the FLR effect.

Equation (17) simplifies when the spatial variations are limited to a direction making a fixed angle with the ambient magnetic field. After a simple rescaling, one gets

$$\frac{\partial u}{\partial T} = \widehat{K}_\Xi \left[\left(\sigma + \frac{\partial^2}{\partial \Xi^2}\right) u - 3u^2\right], \tag{18}$$

where Ξ is the coordinate along the direction of variation. This equation can be referred to as a "dissipative Korteveg-de Vries (KdV) equation", since its stationary solutions coincide

with those of the usual KdV equation. The presence of the Hilbert transform in Eq. (18) nevertheless leads to a dynamics significantly different from that described by soliton equations. Besides, it is worth noting also that Eq. (17) in the two-dimensional case has some similarity with the KP equation (see, Section IV).

B. Properties of the asymptotic model

Equation (17) (and its 1D reduction (18) as well) possesses the remarkable property of being of the form

$$\frac{\partial u}{\partial T} = -\widehat{K}_z \frac{\delta F}{\delta u},$$

where

$$\begin{aligned} F &= \int \left[-\frac{\sigma}{2} u^2 + \frac{1}{2} u \Delta_\perp^{-1} \partial_Z^2 u + \frac{1}{2} \left(\nabla_\perp u \right)^2 + u^3 \right] d\mathbf{R} \\ &\equiv -\sigma N/2 + I_1/2 + I_2/2 + I_3 \end{aligned} \tag{19}$$

has the meaning of a free energy or a Lyapunov functional. This quantity can only decrease in time, since

$$\frac{dF}{dt} = \int \frac{\delta F}{\delta u} \frac{\partial u}{\partial t} d\mathbf{R} = -\int \frac{\delta F}{\delta u} \widehat{K}_z \frac{\delta F}{\delta u} d\mathbf{R} \le 0. \tag{20}$$

This derivative can only vanish at the stationary localized solutions, defined by the equation

$$\frac{\delta F}{\delta u} = \left(\sigma - \Delta_\perp^{-1} \frac{\partial^2}{\partial Z^2} + \Delta_\perp \right) u - 3u^2 = 0. \tag{21}$$

We now show that non-zero solutions of this equation do not exist above threshold ($\sigma = +1$). For this aim, following Ref. [38], we establish relations between the integrals N, I_1, I_2 and I_3, using the fact that solutions of Eq. (21) are stationary points of the functional F (i.e. $\delta F = 0$). Multiplying Eq. (21) by U and integrating over $\mathbf{R}$ gives the first relation

$$\sigma N - I_1 - I_2 - 3I_3 = 0.$$

Two other relations can be found if one makes the scaling transformations, $Z \to aZ$, $\mathbf{R}_\perp \to b\mathbf{R}_\perp$, under which the free energy (19) becomes a function of two scaling parameters a and b

$$F(a,b) = -\frac{\sigma N}{2} ab^2 + \frac{I_1}{2} b^4 a^{-1} + \frac{I_2}{2} a + I_3 ab^2.$$

Due to the condition $\delta F = 0$, the first derivatives of F at $a = b = 1$ have to vanish:

$$\frac{\partial F}{\partial a} = -\frac{\sigma N}{2} - \frac{I_1}{2} + \frac{I_2}{2} + I_3 \; = \; 0,$$
$$\frac{\partial F}{\partial b} = -\sigma N + 2I_1 + 2I_3 \; = \; 0.$$

Hence, after simple algebra, one gets the three relations

$$I_1 + \frac{\sigma}{2} N = 0, \;\; I_3 = -2I_1, \;\; I_2 = 3I_1.$$

For $\sigma = +1$, the first relation can be satisfied only by the trivial solution $u = 0$, because both integrals I_1 and N are positive definite. In other words, above threshold, nontrivial stationary solutions obeying the prescribed scalings do not exist.

In contrast, below threshold, stationary localized solutions can exist. For these solutions, the free energy is positive and reduces to $F_s = N/2$. Furthermore, $I_3 = \int U^3 d^3R < 0$ which means that the structures have the form of magnetic holes. As stationary points of the functional F, these solutions represent saddle points, since the corresponding determinant of second derivatives of F with respect to scaling parameters taken at these solutions is negative ($\partial_{aa} F \partial_{bb} F - (\partial_{ab} F)^2 = -2N^2 < 0$). As a consequence, there exist directions in the eigenfunction space for which the free-energy perturbation is strictly negative, corresponding to linear instability of the associated stationary structure. This is one of the properties for subcritical bifurcations.

As a consequence, starting from general initial conditions, the derivative dF/dt (20) is almost always *negative*, except for unstable stationary points (zero measure) below threshold. In the nonlinear regime, negativeness of this derivative implies $\int u^3 d^3R < 0$, which corresponds to the formation of magnetic holes. Moreover, this nonlinear term (in F) is responsible for collapse, i.e. formation of singularity in a finite time.

C. Blow-up

In order to characterize the nature of the singularity of Eq. (18), it is convenient to introduce the similarity variables $\xi = (T_0 - T)^{-1/3}\Xi$, $\tau = -\log(T_0 - T)$, and to look for a solution in the form $U = (T_0 - T)^{-2/3}\, g(\xi, \tau)$, where $g(\xi, \tau)$ satisfies the equation

$$\frac{\partial g}{\partial \tau} + \frac{2}{3} g + \frac{\xi}{3} \frac{\partial g}{\partial \xi} = \widehat{K}_\xi \left[\frac{\partial^2 g}{\partial \xi^2} - 3g^2 \right] + e^{-\tau} \widehat{K}_\xi g.$$

As time T approaches T_0 ($\tau \to \infty$), the last term in this equation becomes negligibly small and simultaneously $\partial_\tau g \to 0$ so that asymptotically the equation transforms into

$$\frac{2}{3}g + \frac{\xi}{3}\frac{dg}{d\xi} = \widehat{K}_\xi\left[\frac{d^2 g}{d\xi^2} - 3g^2\right]. \tag{22}$$

For the free energy this means that close to T_0 the first term $\sim N$ turns out to be much smaller in comparison with all other contributions, in particular with $\int U^3 d\Xi$.

At large $|\xi|$, that corresponds to the limit $T \to T_0$, the asymptotic solution $\widetilde{g}$ of Eq. (22) obeys

$$2\widetilde{g} + \xi\frac{d\widetilde{g}}{d\xi} = C\xi^{-2}$$

where $C = \frac{9}{\pi}\int_{-\infty}^{\infty} g^2(\xi')d\xi' > 0$, and has the form $\widetilde{g} = C\xi^{-2}\log|\xi/\xi_0|$. For U, it gives the asymptotic solution

$$U_{asymp} = \frac{C}{\Xi^2}\log|\Xi/\Xi_0(t)|$$

with $\Xi_0(t) = (T_0 - T)^{1/3}\xi_0$, that, as $T \to T_0$, has an almost time independent tail. For $|\Xi| < (T_0 - T)^{1/3}|\xi_0|$, the solution is negative and becomes singular as Ξ approaches the origin.

Asymptotically self-similar solutions can also be constructed in three dimensions, when rescaling the longitudinal coordinate by $(T_0 - T)^{1/2}$, the transverse ones by $(T_0 - T)^{1/4}$ and the amplitude of the solution by $(T_0 - T)^{-1/2}$. Existence of a finite time singularity for the initial value problem can be established for initial conditions for which the functional F is negative, when the term involving σ can be neglected, an approximation consistent with the dynamics:

$$F \to F_{\rm lim} \equiv \frac{I_1}{2} + \frac{I_2}{2} + I_3. \tag{23}$$

To prove this statement, consider the operator $\widehat{K}_z^{-1}$, (inverse of the operator $\widehat{K}_z$), which is defined on functions obeying $\int U(Z,\mathbf{R}_\perp)dZ = 0$, a condition consistent with Eq. (17). Then the time derivative of $F_{\rm lim}$ can be rewritten through the operator $\widehat{K}_z^{-1}$ as follows,

$$\frac{dF_{\rm lim}}{dT} = -\int U_T \widehat{K}_z^{-1} U_T d\mathbf{R} \leq 0. \tag{24}$$

Consider now the positive definite quantity $\widetilde{N} = \int U\widehat{K}_z^{-1}U d\mathbf{R} \geq 0$, whose dynamics is determined by the equation

$$\frac{d\widetilde{N}}{dT} = -2\left(I_1 + I_2 + 3I_3\right) = -6F_{\rm lim} + I_1 + I_2. \tag{25}$$

Let $F_{\rm lim}$ be negative initially, then at $T \geq 0$ the r.h.s. of (25) will be positive, and, as a consequence, $\widetilde{N}$ will be a growing function of time.

Introduce now the new quantity $S = -F_{lim}/\widetilde{N}$ which is positive definite if $F_{lim}|_{T=0} < 0$. The time derivative of S is then defined by means of Eqs. (24) and (25):

$$\frac{dS}{dT} = -\frac{F_{\rm lim}\widetilde{N}_T}{\widetilde{N}^2} + \frac{1}{\widetilde{N}} \int U_T \widehat{K}_z^{-1} U_T d\mathbf{R}. \tag{26}$$

The second term in the r.h.s. of this equation can be estimated using the Cauchy-Bunyakowsky inequality:

$$\frac{d\widetilde{N}}{dT} = 2 \int U \widehat{K}_z^{-1} U_T d\mathbf{R} \leq 2\widetilde{N}^{1/2} \left(\int U_T \widehat{K}_z^{-1} U_T d\mathbf{R} \right)^{1/2},$$

that gives

$$\int U_T \widehat{K}_z^{-1} U_T d\mathbf{R} \geq \widetilde{N}_T^2/(4\widetilde{N}).$$

Substituting the obtained estimate into Eq. (26) and taking into account definition (23) for $F_{\rm lim}$ and Eq. (25) as well, we arrive at the differential inequality for S (compare with [39]):

$$\frac{dS}{dT} \geq \frac{\widetilde{N}_T}{\widetilde{N}^2} \left[\frac{\widetilde{N}_T}{4} - F_{\rm lim} \right] \geq 15\, S^2.$$

Integrating this first-order differential inequality yields

$$S \geq \frac{1}{15(T_0 - T)}. \tag{27}$$

Here the collapse time $T_0 = (15\, S_0)^{-1}$ is expressed in terms of the initial value $S|_{t=0} = S_0$. It is interesting to mention that the time behavior of S given by the estimate (27) coincides with that given by the self-similar asymptotics.

D. Conclusion of Section III

We have presented an asymptotic description of the nonlinear dynamics of mirror modes near the instability threshold. Below threshold, we have demonstrated the existence of unstable stationary solutions. Differently, above threshold, no stationary solution consistent with the prescribed small-amplitude, long-wavelength scaling can exist. For small-amplitude initial conditions, the time evolution predicted by the asymptotic equation (17) leads to a finite-time singularity. These properties are based on the fact that this equation belongs

to the generalized gradient systems for which it is possible to introduce a free energy or a Lyapunov functional that decreases in time.

The singularity formation as well as the existence of unstable stationary structures below the mirror instability threshold obtained with the asymptotic model, can be viewed as features of a subcritical bifurcation towards a large-amplitude state that cannot be described in the framework of the present analysis. Such an evolution should indeed involve saturation mechanisms that become relevant when the perturbation amplitudes become comparable with the ambient field.

IV. ADIABATIC APPROACH: ACCOUNT OF ELECTRONS

The mirror instability, as known, is a kinetic instability whose growth rate was first obtained under the assumption of cold electrons [4], a regime where the contributions of the parallel electric field $E_\parallel$ can be neglected. However, in realistic space plasmas, the electron temperature can hardly be ignored [42]. The linear theory retaining the electron temperature and its possible anisotropy, in the quasi-hydrodynamic limit (which neglects finite Larmor radius corrections), was developed in the case of bi-Maxwellian distribution functions by several authors (see e.g. [43], [9], [10]). A general estimate of the growth rate under the sole condition that it is small compared with the ion gyrofrequency (a condition reflecting close vicinity to threshold) is presented in [24]. Like for the cold electrons case, the instability develops in quasi-perpendicular directions, making the parallel magnetic perturbation dominant. This analysis includes in particular regimes with a significant electron temperature anisotropy for which the instability extends beyond the ion Larmor radius. In the limit where the instability is limited to scales large compared with the ion Larmor radius ρ_i, only the leading order contribution in terms of the small parameter $\gamma/(|k|_z v_{\parallel i})$ is to be retained in estimating Landau damping, and the growth rate is given by

$$\begin{aligned}\gamma = \frac{2}{\sqrt{\pi}}\frac{T_{\parallel i}}{T_{\perp i}}\frac{|k_z|v_{\parallel i}}{E}\Big\{\Gamma - \frac{1}{\beta_\perp}\Big(1+\frac{\beta_\perp-\beta_\parallel}{2}\Big)\frac{k_z^2}{k_\perp^2}\\ -\frac{3}{4(1+\theta_\perp)}\Big(\frac{T_{\perp i}}{T_{\parallel i}}-1\Big)(1+F)k_\perp^2 r_L^2\Big\},\end{aligned} \tag{28}$$

where

$$\Gamma = \frac{T_{\perp i}}{T_{\parallel i}}\frac{(\theta_\parallel+\theta_\perp)^2+2\theta_\parallel(\theta_\perp^2+1)}{2\theta_\parallel(1+\theta_\perp)(\theta_\parallel+1)} - 1 - \frac{1}{\beta_\perp} \tag{29}$$

measures the distance to threshold and

$$
\begin{aligned}
E &= \frac{1+\theta_\perp}{(1+\theta_\|)^2}\left[2+\theta_\perp(4+\theta_\perp)+\theta_\|^2\right] \\
F &= \frac{T_{\|e}}{T_{\|e}+T_{\|i}}\Big\{-1+\frac{\theta_\perp}{\theta_\|} \\
&\quad -\frac{2}{3}\frac{T_{\|i}}{T_{\perp i}}\Big[\Big(\frac{T_{\|i}}{T_{\perp i}}-1\Big)\frac{1}{\beta_{\perp i}}-\theta_\perp\Big(\frac{T_{\perp e}}{T_{\|e}}-1\Big)\Big]\Big\}.
\end{aligned}
$$

Here, $T_{\perp\alpha}$ and $T_{\|\alpha}$ are the perpendicular and parallel (relative to the ambient magnetic field $\mathbf{B}_0$ taken in the z direction) temperatures of the species α ($\alpha = i$ for ions and $\alpha = e$ for electrons), $\theta_\perp = T_{\perp e}/T_{\perp i}$, $\theta_\| = T_{\|e}/T_{\|i}$ and $\beta_\perp = \beta_{\perp i}+\beta_{\perp e}$ with $\beta_{\perp\alpha} = 8\pi p_{\perp\alpha}/B_0^2$ where $p_{\perp\alpha}$ is the perpendicular thermal pressure (similar definition for $\beta_\|$). Furthermore, the parallel thermal velocity is defined as $v_{\|\alpha} = \sqrt{2T_{\|\alpha}/m_\alpha}$, and $\rho_i = (2T_{\perp i}/m_p)^{1/2}/\Omega_i$ denotes the ion Larmor radius ($\Omega_i = eB_0/m_ic$ is the ion gyrofrequency).

The growth rate given by Eq. (28) has the same structure as in the cold electron regime considered in the previous sections, and given first time in [13] in the case of bi-Maxwellian ions and then generalized in [9] and [10] to an arbitrary distribution function. The first term within the curly brackets provides the threshold condition which coincides with that given in [43],[13],[44]. The second one reflects the magnetic field line elasticity and the third one (where F depends on the electron temperatures due to the coupling between the species induced by the parallel electric field which is relevant for hot electrons) provides the arrest of the instability at small scales by finite Larmor radius (FLR) effects.

A. Asymptotic model for hot electrons

Now we extend to hot electrons the weakly nonlinear analysis developed for cold electrons in the previous section. Since in this asymptotics, FLR contributions appear only at the linear level, the idea is to use the drift kinetic formalism to calculate the nonlinear terms. We show that the equation governing the evolution of weakly nonlinear mirror modes has the same form as in the case of cold electrons. In particular, the sign of the nonlinear coupling coefficient that prescribes the shape of mirror structures, is not changed, in the case of bi-Maxwellian distributions for both electrons and ions, but can be changed for another distributions. This equation is of gradient type with a free energy (or a Lyapunov functional) which is unbounded from below. This leads to finite-time blowing-up solutions [49, 50],

associated with the existence of a subcritical bifurcation [17, 18]. To describe subcritical stationary mirror structures in the strongly nonlinear regime, we present an anisotropic MHD model where the perpendicular and parallel pressures are determined from the drift kinetic equations in the adiabatic approximation, in the form of prescribed functions of the magnetic field amplitude only.

A main condition governing the nonlinear behavior of mirror modes is provided by the force balance equation

$$-\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \left[1 + \frac{4\pi}{B^2}(p_\perp - p_\|)\right]\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}$$
$$+\mathbf{B}(\mathbf{B}\cdot\nabla)\left(\frac{p_\perp - p_\|}{B^2}\right) - \nabla\cdot\mathbf{\Pi} = 0, \tag{30}$$

where a gyroviscous contribution $\mathbf{\Pi}$ originating from FLR effects (compare with (17)). Note that FLR contributions also enter the gyrotropic pressures. Here the pressure tensor and its components are viewed as the sum of the contributions of the various species. In particular $p_\perp = \sum_\alpha p_{\perp\alpha}$ and $p_\| = \sum_\alpha p_{\|\alpha}$. When concentrating on scales large compared with the electron Larmor radius, the non-gyrotropic correction $\mathbf{\Pi}$ to the pressure tensor originates only from the ions. As mentioned above, it is enough to retain this contribution only at the linear level with respect to the amplitude of the perturbations. As in the case of cold electrons, the other linear and nonlinear contributions can be evaluated from the drift kinetic equation

$$\frac{\partial f_\alpha}{\partial t} + v_\|\mathbf{b}\cdot\nabla f_\alpha + \left(-\mu\mathbf{b}\cdot\nabla B + \frac{e_\alpha}{m_\alpha}E_\|\right)\frac{\partial f_\alpha}{\partial v_\|} = 0 \tag{31}$$

for each type of particles.

We ignore the transverse electric drift which is subdominant for mirror modes. In this approximation, both ions and electrons move in the direction of the magnetic field under the effect of the magnetic force $\mu\,\mathbf{b}\cdot\nabla B$ and the parallel electric field $E_\| = -\mathbf{b}\cdot\nabla\phi$ where the magnetic moment $\mu = v_\perp^2/(2B)$ is an adiabatic invariant which plays the role of a parameter in Eq. (31). Here ϕ is the electric potential. The quasi-neutrality condition $n_e = n_i \equiv n$, where $n_\alpha = B\int f_\alpha d\mu dv_\| d\varphi \equiv \int f_\alpha d^3v$, is used to close the system and eliminate $E_\|$.

In this framework where FLR corrections are neglected, the gyrotropic pressures $p_{\|\alpha}$ and $p_{\perp\alpha}$ are given in terms of the corresponding distribution functions f_α by

$$p_{\alpha\|} = m_\alpha B\int v_\|^2 f_\alpha d\mu dv_\| d\varphi \equiv m_\alpha\int v_\|^2 f_\alpha d^3v,$$
$$p_{\alpha\perp} = m_\alpha B^2\int \mu f_\alpha d\mu dv_\| d\varphi \equiv \frac{1}{2}m_\alpha\int v_\perp^2 f_\alpha d^3v.$$

The equation governing the mirror dynamics is obtained perturbatively by expanding Eqs. (30), (31) and the quasi-neutrality condition. In this approach, the pressure tensor elements for each species are computed near a bi-Maxwellian equilibrium state characterized by temperatures $T_{\perp\alpha}$ and $T_{\|\alpha}$ and a constant ambient magnetic field $\mathbf{B_0}$ taken along the z-direction.

B. Linear instability

Before turning to the nonlinear regime, we briefly reformulate the linear theory in the framework of the drift kinetic approximation, in order to specify the notations.

From Eq. (30), linearized about the background field $\mathbf{B_0}$ by writing $\mathbf{B} = \mathbf{B}_0 + \widetilde{\mathbf{B}}$ ($B_0 \gg \widetilde{B}$) with $\widetilde{\mathbf{B}} \sim e^{-i\omega t + i\,\mathbf{k}\cdot\mathbf{r}}$, we arrive at Eq. (6), where $p_\perp^{(1)}$ has to be calculated from the linearized drift kinetic equation (31) after elimination of the parallel electric field using the quasi-neutrality condition. Note that as for the case of cold electrons, near the instability threshold the leading terms in (6) corresponding to perturbations of perpendicular and magnetic pressures are compensated by each other and therefore one needs to retain the next order terms responsible for both elasticity of magnetic field lines and FLR corrections.

The linearized drift kinetic equation reads

$$\frac{\partial f_\alpha^{(1)}}{\partial t} + v_\| \frac{\partial f_\alpha^{(1)}}{\partial z} + \left(-\mu \frac{\partial \widetilde{B}_z}{\partial z} + \frac{e_\alpha}{m_\alpha} E_\| \right) \frac{\partial f_\alpha^{(0)}}{\partial v_\|} = 0, \tag{32}$$

where we assume each $f_\alpha^{(0)}$ to be a bi-Maxwellian distribution function

$$f_\alpha^{(0)} = A_\alpha \exp\left[-\frac{v_\|^2}{v_{\|\alpha}^2} - \frac{\mu B_0 m_\alpha}{T_{\perp\alpha}} \right], \tag{33}$$

with $A_\alpha = n_0 m_\alpha / (2\pi\sqrt{\pi} v_{\|\alpha} T_{\perp\alpha})$.

In Fourier representation, Eq. (32) is solved as

$$f_\alpha^{(1)} = -\frac{\mu \widetilde{B}_z + \frac{e_\alpha}{m_\alpha}\phi}{\omega - k_z v_\|} k_z \frac{\partial f_\alpha^{(0)}}{\partial v_\|}. \tag{34}$$

The neutrality condition in the linear approximation reads

$$\int f_i^{(1)} dv_\| d\mu d\varphi = \int f_e^{(1)} dv_\| d\mu d\varphi, \tag{35}$$

that allows one to express the potential ϕ in terms of $\widetilde{B}_z$. We have

$$\int f_i^{(1)} dv_z d\mu d\varphi = -\frac{n_0}{B_0 T_{\|i}} \Big[T_{\perp i} \frac{\widetilde{B}_z}{B_0} + e\phi \Big] \Big[1 + \frac{i\sqrt{\pi}\omega}{|k_z| v_{\|i}} \Big]. \tag{36}$$

Here we assume that $\omega/k_z \ll v_{\|i} = \sqrt{2T_{\|i}/m_i}$, so that the contribution from the Landau pole is small ($\xi = \sqrt{\pi}\omega/(|k_z|v_{\|i}) \ll 1$).

An analogous calculation for the electrons, neglecting the contribution of the corresponding Landau resonance because of the small mass ratio (assuming the ratio of the electron to ion temperatures is not too small), gives

$$\int f_e^{(1)} dv_z d\mu d\varphi = -\frac{n_0}{B_0 T_{\|e}} \left(T_{\perp e} \frac{\widetilde{B}_z}{B_0} - e\phi \right). \tag{37}$$

The quasi-neutrality condition then reads

$$\frac{1}{T_{\|i}} \Big[T_{\perp i} \frac{\widetilde{B}_z}{B_0} + e\phi \Big] \Big[1 + \frac{i\sqrt{\pi}\omega}{|k_z|v_{\|i}} \Big] = \frac{1}{T_{\|e}} \Big[T_{\perp e} \frac{\widetilde{B}_z}{B_0} - e\phi \Big], \tag{38}$$

and leads to the estimate

$$e\phi \approx \frac{T_{\perp i}}{1+\theta_\|} \Big[(\theta_\perp - \theta_\|) - \frac{\theta_\|(1+\theta_\perp)}{1+\theta_\|} i\xi \Big] \frac{\widetilde{B}_z}{B_0}. \tag{39}$$

Thus, for cold electrons ($\theta_\perp = \theta_\| = 0$), ϕ vanishes and the influence of the parallel electric field on the mirror instability becomes negligible. Interestingly, when $\theta_\perp = \theta_\|$, only the Landau pole contributes to

$$e\phi \approx -\frac{T_{\perp i}\theta_\|}{1+\theta_\|} i\xi \frac{\widetilde{B}_z}{B_0}. \tag{40}$$

Now, it is necessary to evaluate

$$p_\perp^{(1)} = 2\frac{\widetilde{B}_z}{B_0} p_\perp^{(0)} + B_0^2 \sum_\alpha m_\alpha \int \mu f_\alpha^{(1)} d\mu dv_\| d\varphi. \tag{41}$$

Using

$$\begin{aligned} \int \frac{k_z v_\|}{\omega - k_z v_\|} f_i^{(0)} d\mu dv_\| d\varphi &= -\frac{n_0}{B_0}(1+i\xi) \\ \int \frac{k_z v_\|}{\omega - k_z v_\|} f_e^{(0)} d\mu dv_\| d\varphi &= -\frac{n_0}{B_0}, \end{aligned}$$

we get

$$\sum_\alpha m_\alpha \int \mu f_\alpha^{(1)} d\mu dv_\| d\varphi \approx -n_0 \frac{T_{\perp i}^2}{T_{\|i}} \frac{\widetilde{B}_z}{B_0^3} (C + i\xi D), \tag{42}$$

where the coefficients C and D, defined above, are both positive. In the cold-electron limit, $C \to 2$ and $D \to 2$.

It is worth noting that the terms $-\frac{\big(\theta_\perp - \theta_\|\big)^2}{\theta_\|(1+\theta_\|)}$ in C and $\big(\theta_\perp - \theta_\|\big) \frac{\big(2+\theta_\perp+\theta_\|\big)}{\big(1+\theta_\|\big)^2}$ in D originate from the contributions of the electrostatic potential ϕ to $p_\perp^{(1)}$, and vanish for

$\theta_\perp = \theta_\parallel$. Furthermore, in this limit, the real part of the perpendicular pressure fluctuations is the sum of two independent contributions originating from the ions and the electrons. Differently, only the ion Landau pole contribution is retained in the imaginary part. We finally get

$$p_\perp^{(1)} = \frac{\widetilde{B}_z}{B_0} n_0 T_{\perp i} \left[2(1+\theta_\perp) - \frac{T_{\perp i}}{T_{\parallel i}} (C + i\xi D) \right]. \tag{43}$$

Substituting this expression into the linearized force balance equation yields

$$\begin{aligned} n_0 T_{\perp i} \frac{T_{\perp i}}{T_{\parallel i}} D i\xi &= 2 n_0 T_{\perp i}(1+\theta_\perp) \\ &\times \left[1 - \frac{T_{\perp i}}{2T_{\parallel i}(1+\theta_\perp)} C + \frac{1}{\beta_\perp} + \frac{k_z^2}{k_\perp^2 \beta_\perp}\chi \right], \end{aligned}$$

and thus the linear growth rate

$$\begin{aligned} \gamma &= |k_z| v_{\mathrm{th}\parallel i} \frac{2}{\sqrt{\pi}} \frac{T_{\parallel i}}{T_{\perp i}} \frac{1+\theta_\perp}{D} \\ &\times \left[\frac{T_{\perp i}}{2T_{\parallel i}(1+\theta_\perp)} C - 1 - \frac{1}{\beta_\perp} - \frac{k_z^2}{k_\perp^2 \beta_\perp}\chi \right], \end{aligned} \tag{44}$$

where $\chi = 1 + (\beta_\perp - \beta_\parallel)/2$. It reproduces Eq. (28) up to the FLR term which is not captured by the drift kinetic approximation. As $\theta \to 0$, the growth rate reduces to the usual form given in [4]

$$\gamma = |k_z| v_{i\parallel} \frac{\beta_\parallel}{\sqrt{\pi}\beta_\perp} \left[\frac{\beta_\perp}{\beta_\parallel} - 1 - \frac{1}{\beta_\perp} - \frac{k_z^2}{k_\perp^2 \beta_\perp}\chi \right]. \tag{45}$$

In the presence of hot electrons, the mirror instability arises when

$$\begin{aligned} \Gamma = &\frac{T_{\perp i}}{2T_{\parallel i}(1+\theta_\perp)} \frac{1}{\theta_\parallel(\theta_\parallel+1)} \left[(\theta_\parallel + \theta_\perp)^2 + 2\theta_\parallel(\theta_\perp^2 + 1) \right] \\ &- 1 - \frac{1}{\beta_\perp} > 0 \end{aligned} \tag{46}$$

and, near threshold, develops in quasi-perpendicular directions, making the parallel magnetic perturbation dominant. This instability condition can be also rewritten in the form given in [43].

Note that the growth rate derived above is valid provided the condition $\gamma/k_z \ll v_{\mathrm{th}\parallel i}$ is fulfilled. Furthermore, the instability is arrested by FLR effects at scales that are too small to be captured by the drift kinetic asymptotics.

C. General pressure estimates

As demonstrated in Section 2 (see also [17, 18]), the scalings (11) resulting from the linear theory near threshold, when k_z and $k_\perp$ vary proportionally to ε and $\sqrt{\varepsilon}$ respectively, while the instability growth rate behaves like $\sim \varepsilon^2$, imply an adiabaticity condition, or, in another words, this leads to the stationary kinetic equation

$$v_\| \mathbf{b}\cdot\nabla f_\alpha - (\mathbf{b}\cdot\nabla)\Big[\mu B + \frac{e_\alpha}{m_\alpha}\phi\Big]\frac{\partial f_\alpha}{\partial v_\|} = 0. \tag{47}$$

It in fact turns out that Eq. (47) is exactly solvable, the general solution being an arbitrary function of all integrals of motion $f_\alpha = g_\alpha(\mu, W_\alpha, q)$ of the particle energy $W_\alpha = \frac{v_\|^2}{2} + \mu B + \frac{e_\alpha}{m_\alpha}\phi$, of μ and of variables q responsible for labeling the magnetic field lines. As we see in the previous case on the example of cold electrons, the dependence on q does not appear in the weakly nonlinear regime, analyzed perturbatively. In the next section, we will return to this question and discuss it in more detail. Below we will ignore this dependence, considering only the case when f_α has two arguments μ and W_α.

To find the function $g_\alpha(\mu, W_\alpha)$ in this case, we use the adiabaticity argument which means that, to leading order, g_α as a function of its arguments μ and W_α retains its form during the evolution. Therefore, the function $g_\alpha(\mu, W_\alpha)$ is found by matching with the initial distribution function $f_\alpha^{(0)}$, given by Eq. (33) and corresponding to $\phi = 0$ and $W_\alpha = \frac{v_\|^2}{2} + \mu B_0$. We get

$$\begin{aligned} g_\alpha(\mu, W_\alpha) &= A_\alpha \exp\Big[-\frac{v_\|^2}{v_{\|\alpha}^2} - \frac{\mu B_0 m_\alpha}{T_{\perp\alpha}}\Big] \\ &= A_\alpha \exp\Big[-\frac{2}{v_{\|\alpha}^2}\Big(\frac{v_\|^2}{2} + \mu B_0\Big) \\ &\qquad + \mu B_0 m_\alpha\Big(\frac{1}{T_{\|\alpha}} - \frac{1}{T_{\perp\alpha}}\Big)\Big] \\ &= A_\alpha \exp\Big[-\frac{2W_\alpha}{v_{\|\alpha}^2} + \mu B_0 m_\alpha\Big(\frac{1}{T_{\|\alpha}} - \frac{1}{T_{\perp\alpha}}\Big)\Big]. \end{aligned} \tag{48}$$

Thus, $g_\alpha(\mu, W_\alpha)$ is a Boltzmann distribution function with respect to W_α but, at fixed W_α, it displays an exponential growth relatively to μ if $T_{\perp\alpha} > T_{\|\alpha}$. This effect can however be compensated by the dependence of W_α in μ. This means that only a fraction of the phase space (μ, W_α) is accessible, a property possibly related with the concepts of trapped and untrapped particles.

Note that expanding Eq. (48) relatively to $\widetilde{B}_z$ and $\phi^{(1)}$ reproduces the first order contribution to the distribution function (34) with $\omega = 0$ and also the corresponding expression for the second order correction (15) found in the previous section (see also [17], [18]) in the case of cold electrons. It should be emphasized that Eq. (48) only assumes adiabaticity and remains valid for finite perturbations.

The function g_α can also be rewritten in terms of $v_\|$, $v_\perp$ and ϕ as

$$g_\alpha = A_\alpha \exp\Big[-\frac{m_\alpha v_\|^2}{2T_{\|\alpha}} - \frac{e_\alpha \phi}{T_{\|\alpha}}\Big] \times$$
$$\exp\left\{-\frac{m_\alpha v_\perp^2}{2T_{\perp\alpha}}\Big(\frac{T_{\perp\alpha}}{T_{\|\alpha}} - \frac{B_0}{B}\Big[\frac{T_{\perp\alpha}}{T_{\|\alpha}} - 1\Big]\Big)\right\},$$

which can be viewed as the bi-Maxwellian distribution function with the renormalized transverse temperature

$$T_{\perp\alpha}^{(eff)} = T_{\perp\alpha}\left[\frac{T_{\perp\alpha}}{T_{\|\alpha}} - \frac{B_0}{B}\Big(\frac{T_{\perp\alpha}}{T_{\|\alpha}} - 1\Big)\right]^{-1}. \tag{49}$$

Note the Boltzmann factor $\exp -[e_\alpha\phi/T_{\|\alpha}]$ in the expression of g_α. For cold electrons, the ion distribution function was obtained in [51] by assuming that the distribution remains bi-Maxwellian and owing to the invariance of the kinetic energy and of the magnetic moment. This estimate obtained by neglecting both time dependency (and consequently the Landau resonance) and finite Larmor radius corrections reproduces the closure condition given in [8].

After rewriting Eq. (48) in the form

$$g_\alpha = A_\alpha \exp\Big[-\frac{e_\alpha\phi}{T_{\|\alpha}} - \frac{v_\|^2}{v_{\|\alpha}^2} - \frac{\mu B_0 m_\alpha}{T_{\perp\alpha}}\Big(1 + \frac{T_{\perp\alpha}}{T_{\|\alpha}}\frac{B - B_0}{B_0}\Big)\Big], \tag{50}$$

the quasi-neutrality condition gives

$$\left(1 + \frac{T_{\perp i}}{T_{\|i}}\frac{B - B_0}{B_0}\right)^{-1} \exp\left(-\frac{e\phi}{T_{\|i}}\right) =$$
$$\left(1 + \frac{T_{\perp e}}{T_{\|e}}\frac{B - B_0}{B_0}\right)^{-1} \exp\left(\frac{e\phi}{T_{\|e}}\right)$$

or

$$e\phi = (T_{\|i}^{-1} + T_{\|e}^{-1})^{-1} \times$$
$$\log\left[\left(1 + \frac{T_{\perp e}}{T_{\|e}}\frac{B - B_0}{B_0}\right)\left(1 + \frac{T_{\perp i}}{T_{\|i}}\frac{B - B_0}{B_0}\right)^{-1}\right]. \tag{51}$$

Interestingly, the electron density

$$n_e = n_0 \frac{B}{B_0} \left(1 + \frac{T_{\perp e}}{T_{\parallel e}} \frac{B - B_0}{B_0} \right)^{-1} \exp \left[\frac{e\phi}{T_{\parallel e}} \right] \tag{52}$$

has the usual Boltzmann factor $\exp \left[e\phi / T_{\parallel e} \right]$ and also an algebraic prefactor depending on the magnetic field B. In the case of isotropic electron temperature ($T_{\perp e} = T_{\parallel e} \equiv T_e$), the electron density has the usual Boltzmann form $n_e = n_0 \exp \left[e\phi / T_e \right]$.

The above formula for ϕ shows that the potential vanishes in two cases: for cold electrons and when electron and ion temperature anisotropies a_e and a_i are equal, a case first time mentioned in the linear theory of the mirror instability [12, 13, 43].

Equation (51) allows one to evaluate explicitly the perpendicular pressure for each species

$$\begin{aligned} p_{\perp \alpha} &= m_\alpha B^2 \int \mu g_\alpha d\mu dv_\parallel d\varphi \\ &= n_0 T_{\perp \alpha} \frac{B^2}{B_0^2} \Big(1 + \frac{T_{\perp \alpha}}{T_{\parallel \alpha}} \frac{B - B_0}{B_0} \Big)^{-2} \exp \Big(- \frac{e_\alpha \phi}{T_{\parallel \alpha}} \Big), \end{aligned}$$

where $e\phi$ is given by Eq. (51).

Hence, simple algebraic procedure gives the following expression for the parallel pressure [24], [45]:

$$p_\parallel = n_0 (T_{\parallel i} + T_{\parallel e}) \frac{1 + u}{(1 + a_e u)^{c_e} (1 + a_i u)^{c_i}}, \tag{53}$$

where $u = B/B_0 - 1$, $a_\alpha = T_{\perp \alpha} / T_{\parallel \alpha}$ is the parameter characterizing the anisotropy of distribution function f_α, and $c_\alpha = T_{\parallel \alpha} (T_{\parallel e} + T_{\parallel i})^{-1}$ in the case of a proton-electron plasma. As it will be shown in the next section, the perpendicular pressure can be easily found by means of the general relation

$$p_\perp = p_\parallel - B \frac{dp_\parallel}{dB}. \tag{54}$$

Substitution of (53) into this expression yeilds

$$p_\perp = p_\parallel (1 + u) \left(\frac{c_e a_e}{1 + a_e u} + \frac{c_i a_i}{1 + a_i u} \right). \tag{55}$$

Hence one can see that both pressures have the singularities at $u = -a_\alpha^{-1}$ corresponding to the magnetic field

$$B_s = B_0 \frac{a_\alpha - 1}{a_\alpha} < B_0. \tag{56}$$

In the limiting case of cold electrons, $p_\parallel = n_0 T_\parallel (1 + u)(1 + au)^{-1}$ displays a pole singularity. Here, $T_\parallel$ and the anisotropy parameter a correspond to ions only. Such an equation of state was previously derived by a quasi-normal closure of the fluid hierarchy [8].

The above singularities are presumably related to an overestimated contribution from large μ, corresponding either to small B or to large a transverse kinetic energy. In both cases, the applicability of the drift approximation breaks down and we are thus led to introduce some cut-off type correction near μ^*_α. In a simple variant, we take $f_\alpha = \tilde{C}_\alpha \exp(-m_\alpha W_\alpha / T_{\|\alpha})$ at $\mu > \mu^*_\alpha$, with some positive constant $\tilde{C}_\alpha$, and f_α retains its original form (48) for $\mu \le \mu^*_\alpha$. For cold electrons, the parallel ion pressure is modified into $p_\| = n_0 T_\| G(B,r)$ with

$$G(B,r) = \frac{1}{1+C}\left[\frac{(B_0 - B_s)B}{B_0(B-B_s)} R(B,r) + Ce^{r(B_0-B)}\right],$$

and

$$R(B,r) = \frac{\exp[-r(B-B_s)] - 1}{\exp[-r(B_0-B_s)] - 1}.$$

Here, C is a (small) constant, and $r = m\mu^*/T_\|$. Noticeably, regularization leads to a non-singular positive pressure for all B, including when $B \to 0$. The modification for $p_\|$ in the case of hot electrons is not specified here because the expressions are algebraically much more cumbersome but do not involve any additional difficulty.

After these remarks, one can easily derive the asymptotic model with account of hot electrons. The basic idea is the same as we used already while derivation the model (17) for cold electrons. To derive the asymptotic model, we can of course forget about renormalization of the function $G(B,r)$ because we need to consider the expansion of $p_\perp$ with respect to small amplitude u by taking into account in this expansion only the second term $\sim u^2$ which defines the nonlinear coupling coefficient for (17). For (55) the quadratic contributions originating from $p_\perp^{(2)} + (B-B_0)^2/(8\pi)$ are collected in a term $\Lambda\left(\frac{B-B_0}{B_0}\right)^2$ with

$$\begin{aligned}
\Lambda \;=\; & n_0\Big\{T_{\perp i}\Big(3a_i^2 - 4a_i + 1 \\
& + c_i(a_e - a_i)\Big[\frac{1}{2}(1+c_i)(a_e - a_i) - 2 + 3a_i\Big]\Big) \\
& + T_{\perp e}\Big(3a_e^2 - 4a_e + 1 + c_e(a_e - a_i) \\
& \times\Big[\frac{1}{2}(1+c_e)(a_e - a_i) + 2 - 3a_e\Big]\Big)\Big\} + \frac{B_0^2}{8\pi}.
\end{aligned} \tag{57}$$

The value Λ_c of Λ at threshold is obtained by expressing $\dfrac{B_0^2}{8\pi}$ by means of Eq. (29), which gives

$$\begin{aligned}
\Lambda_c \;=\; & n_0\Big\{T_{\perp i}\Big[3a_i^2 - 4a_i + 1 \\
& + c_i(a_e - a_i)\Big(\frac{1}{2}(1+c_i)(a_e - a_i) - 2 + 3a_i\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\Big(2-2a_i-c_i(a_e-a_i)\Big)\Big]\\
&+T_{\perp e}\Big[3a_e^2-4a_e+1+c_e(a_e-a_i)\\
&\times\Big(\frac{1}{2}(1+c_e)(a_e-a_i)+2-3a_e\Big]\\
&-\frac{1}{2}\Big(2-2a_e+c_e(a_e-a_i)\Big)\Big]\Big\}.
\end{aligned}
$$

After some algebra, one gets

$$
\begin{aligned}
\frac{\lambda_c}{\alpha_i} = \frac{T_{\perp i}}{T_{\| i}}&\Big[3+3\frac{\theta_\perp^3}{\theta_\|^2}-\frac{1}{2}\frac{\left(\theta_\perp-\theta_\|\right)^2}{\theta_\|^2\left(1+\theta_\|\right)^2}\\
&\times\left(4\theta_\perp+4\theta_\|^2+5\left(\theta_\perp+1\right)\theta_\|\right)\Big]\\
&-\frac{3}{2\theta_\|\left(1+\theta_\|\right)}\left[\left(\theta_\perp+\theta_\|\right)^2+2\theta_\|(1+\theta_\perp^2)\right],
\end{aligned}
\tag{58}
$$

where $\lambda_c = \Lambda_c/(n_0 T_{\perp i})$.

Supplementing the corresponding quadratic terms in Eq. (28) leads, at the order of the expansion, to the dynamical equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\widetilde{B}_z}{B_0} &= \frac{2}{\sqrt{\pi}}\frac{T_{\| i}}{T_{\perp i}}\frac{v_{\mathrm{th}\| i}}{D}\left(-\mathcal{H}\partial_z\right)\\
&\times\Big\{\Big[\frac{T_{\perp i}}{T_{\| i}}\frac{C}{2}-(1+\theta_\perp)\Big(1+\frac{1}{\beta_\perp}\Big)\Big]\frac{\widetilde{B}_z}{B_0}\\
&-(1+\theta_\perp)\frac{1}{\beta_\perp}\Big(1+\frac{\beta_\perp-\beta_\|}{2}\Big)(\Delta)^{-1}\partial_{zz}\frac{\widetilde{B}_z}{B_0}\\
&+\frac{3}{4}\Big(\frac{T_{\perp i}}{T_{\| i}}-1\Big)(1+F)r_L^2\Delta_\perp\frac{\widetilde{B}_z}{B_0}-\frac{\lambda_c}{2}\Big(\frac{\widetilde{B}_z}{B_0}\Big)^2\Big\}
\end{aligned}
$$

that extends the result of [17, 37] valid for cold electrons. As demonstrated in [17, 18], the sign of the nonlinear coupling λ_c defines the type of subcritical structures, namely holes ($\lambda_c > 0$) or humps ($\lambda_c < 0$). It turns out that the sign of the nonlinear coupling can be determined analytically in a few special cases.

(i) Limit $\theta_\| \ll \theta_\perp$:

$$
\frac{\Lambda_c}{n_0 T_{\perp i} a_i} = \frac{\theta_\perp^2}{\theta_\|}\left(\frac{T_{\perp e}}{T_{\| e}}-\frac{3}{2}\right) > 0. \tag{59}
$$

(ii) Equal anisotropies ($\theta_\perp = \theta_\|$)

$$
\begin{aligned}
\Lambda_c \;=\; & n_0(T_{\perp i}+T_{\perp e})\left(3a^2-4a+1\right)\\
&-n_0(T_{\perp i}+T_{\perp e})\,(1-a) = 3a\frac{B_0^2}{8\pi} > 0.
\end{aligned}
$$

(iii) Isotropic electron temperature: The coefficient Λ_c can be rewritten in the form

$$\begin{aligned}\Lambda_c &= n_0(a_i-1)\{T_{\perp i}\Big((3a_i-1)\\ &+c_i\Big[\frac{1}{2}(1+c_i)\,(\alpha_i-1)+2-3a_i\Big]\Big)\\ &+T_e c_e\Big[\frac{1}{2}\,(1+c_e)\,(a_i-1)+1\Big]\}+\frac{B_0^2}{8\pi}.\end{aligned}$$

Furthermore, at threshold

$$\frac{1}{2}n_0(a_i-1)\left[T_{\perp i}\,(2-c_i)+T_{\perp e}c_e\right]=\frac{B_0^2}{8\pi}>0. \tag{60}$$

Hence, we simultaneously have two inequalities $a_i>1$ and $T_{\perp e}c_e>T_{\perp i}(c_i-2)$. Therefore,

$$\begin{aligned}\Lambda_c &= n_0(a_i-1)\Big\{T_{\perp i}\Big((3a_i-1)\\ &+c_i\Big[\frac{1}{2}(1+c_i)\Big(a_i-1\Big)+2-3a_i\Big]\Big)\\ &+T_e c_e\Big[\frac{1}{2}(1+c_e)(a_i-1)+1\Big]\Big\}\\ &+\frac{1}{2}n_0(a_i-1)\left[T_{\perp i}\,(2-c_i)+T_{\perp e}c_e\right]\\ &= n_0(a_i-1)\Big\{T_{\perp i}\Big(3a_i(1-c_i)\\ &+c_i\Big[\frac{1}{2}(1+c_i)\,(a_i-1)+\frac{3}{2}\Big]\Big)\\ &+T_e c_e\Big[2+\frac{1}{2}\,(1+c_e)\,(a_i-1)\Big]\Big\},\end{aligned}$$

which is positive because $1-c_i\equiv c_e=\dfrac{1}{1+\theta_\parallel}>0$ and $a_i-1>0$.

(iv) More general conditions: A numerical approach was used. For this purpose it is of interest to display in Fig. 1, for typical values of the parameters taken here as $\theta_\perp=1$, $a_i=1.1$ and $\beta_{\perp i}=10$, the distance to threshold Γ (dashed line) given by Eq. (46) and the non-dimensional nonlinear coupling coefficient $\lambda=\Lambda/(n_0T_{\perp i})$ (solid line), with Λ given by Eq. (57), as a function of $\theta_\parallel$. This graph is typical of the general behavior of these functions and shows that they are both decreasing as $\theta_\parallel$ increases, with λ possibly reaching negative values, but only below threshold. In order to show that the value λ_c, given by Eq. (58), of λ at threshold is positive in a wider range of parameters, we display in Fig. 2, as a function of $\beta_{\perp i}$ for $\theta_\perp=0.2$ (solid line), $\theta_\perp=1$ (dotted line) and $\theta_\perp=5$ (dashed line), the quantity $\min(\lambda_{\rm c})$ obtained after minimizing λ_c in an interval of values of a_p between 0 and $a_{p1}(\beta_{\perp i})$. The latter quantity is arbitrarily defined such that the threshold is obtained for a value of

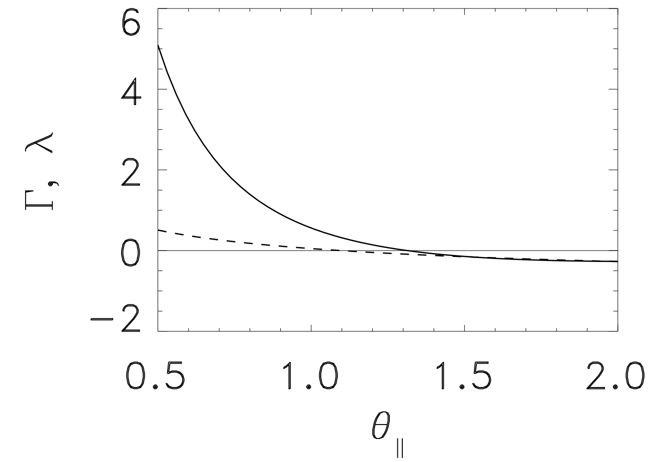

FIG. 1: Variation with $\theta_{\parallel}$ of the distance to threshold Γ given by Eq. (46) (dashed line) and of the normalized nonlinear coupling coefficient λ (solid line) evaluated from Eq. (57) for $\theta_{\perp} = 1$, $a_i = 1.1$ and $\beta_{\perp i} = 10$.

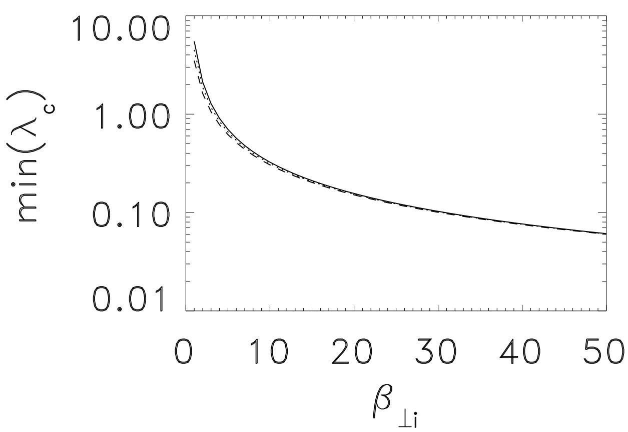

FIG. 2: Variation with $\beta_{\perp i}$ of the minimum $\min(\lambda_c)$ of the normalized nonlinear coupling coefficient taken in an interval of values of a_p between 0 and $a_{p1}(\beta_{\perp i})$, defined such that the threshold is obtained for a value of $\theta_{\parallel}$ equal to 100, for $\theta_{\perp} = 0.2$ (solid line), $\theta_{\perp} = 1$ (dotted line) and $\theta_{\perp} = 5$ (dashed line).

$\theta_{\parallel}$ equal to 100. This graph shows that $\min(\lambda_c)$ varies little with $\theta_{\perp}$ but is very sensitive to $\beta_{\perp i}$. As the latter parameter is increased, $\min(\lambda_c)$ decreases towards zero but remains always positive. Although this numerical observation is definitively not a rigorous proof, it convincingly shows that Λ should remain positive in the parameter regime of physical interest.

Thus, we can see that in the case of the bi-Maxwelian distribution functions for both

ions and electrons (i) the asymptotic model has the same structure as in the cold case, and (ii) it predicts the formation of magnetic holes which is defined by the sign of the coupling coefficient Λ . If the disributions are different for the bi-Maxwellian ones we can expect change of the sign of $\Lambda$and appearance of magnetic structures in the form of humps respectively. In the next sections, we show how such mirror structures can be found for arbitrary distributions for both electrons and ions based on the variational principle when both pressures are functions of the magnetic field amplitude only.

V. VARIATIONAL PRINCIPLE FOR STATIONARY ANISOTROPIC MHD

As we saw in the previous sections, the nonlinearity for mirror modes originates from equations (30) which represent the anisotropic MHD in a static regime supplemented by corrections due to the FLR effects. Secondly, another origin of the nonlinearity comes from the drift kinetic equations, in particular, for the asymptotic model (17) it comes from the stationary kinetic equations (47). Thus, the static anisotropic MHD together with the stationary drift kinetic equations describe the nonlinear development of the mirror modes and its possible saturation in the form of static structures. In this section, we give formulation of the variational principle for such structures and establish connection it with the free energy formalism developed for the asymptotic model (17).

A. Gyrotropic pressure balance

We start from the pressure balance equation for a static gyrotropic MHD equilibrium

$$0 = -\nabla \cdot \mathbf{P} + \frac{1}{c}\left[\mathbf{j} \times \mathbf{B}\right], \tag{61}$$

where the current $\mathbf{j}$ is defined from the Maxwell equation as $\mathbf{j}= \frac{c}{4\pi}\nabla\times\mathbf{B}$, and the pressure tensor $\mathbf{P}$ is assumed to be gyrotropic. The solvability conditions read $\mathbf{B}\cdot(\nabla\cdot\mathbf{P})$=0, and $\mathbf{j}\cdot(\nabla\cdot\mathbf{P})$=0.

In terms of the tension tensor $S_{ij} = \Pi_\perp\left(\delta_{ij} - b_i b_j\right) + S_\parallel b_i b_j$, Eq. (61) takes the divergence form $\frac{\partial}{\partial x_j}\Pi_{ij} = 0$ where $S_\perp = p_\perp + B^2/(8\pi)$ and $S_\parallel = p_\parallel - B^2/(8\pi)$, and the perpendicular and parallel pressures $p_\perp = \Sigma_\alpha p_{\perp\alpha}$ and $p_\perp = \Sigma_\alpha p_{\perp\alpha}$ are the sum of the contributions of the various particle species α. They are expressed as $p_{\perp\alpha} = m_\alpha B^2 \int \mu f_\alpha dv_\parallel d\mu$ and $p_{\parallel\alpha} =$

$m_\alpha B \int v_\parallel^2 f_\alpha dv_\parallel d\mu$, in terms of the distribution functions f_α, which satisfy the stationary drift kinetic equations

$$v_\parallel \nabla_\parallel f_\alpha - \left[\mu \nabla_\parallel B + \frac{e_\alpha}{m_\alpha} \nabla_\parallel \phi\right] \frac{\partial f_\alpha}{\partial v_\parallel} = 0. \tag{62}$$

These equations are supplemented by the quasi-neutrality condition

$$\sum_\alpha e_\alpha B \int f_\alpha dv_\parallel d\mu = 0, \tag{63}$$

that allows one to eliminate the electric potential.

We consider partial solutions of the stationary kinetic equations (62) which are expressed in terms of two integrals of motion: the energy of the particles $W_\alpha = v_\parallel^2/2 + \mu B + (e_\alpha/m_\alpha)\phi$ and their magnetic moment μ. In general, the solution can also depend on integrals which label the magnetic field lines [34]. The choice $f_\alpha = f_\alpha(W_\alpha, \mu)$, as it will be shown further, can be matched with the solution found perturbatively for weakly nonlinear mirror modes within the asymptotic model (17) . In this case, the parallel and perpendicular pressures for the individual species and also the total pressures are functions of B only. We write $p_{\perp\alpha} = p_{\perp\alpha}(B)$ and $p_{\parallel\alpha} = p_{\parallel\alpha}(B)$. As seen in the next subsection, this property plays a very central role in the forthcoming analysis.

B. Identity in the parallel direction and variational principle

According to Ref. [33], the anisotropic pressure balance equation (61)can be easily reformulated as follows:

$$-\nabla p_\parallel - \frac{1}{B}(p_\perp - p_\parallel)\nabla B = \left[\mathbf{B} \times \left[\nabla \times \left(\frac{p_\perp - p_\parallel}{B^2} + \frac{1}{4\pi}\right)\mathbf{B}\right]\right]. \tag{64}$$

Hence projection along the magnetic field gives

$$-\nabla_\parallel p_\parallel - \frac{4\pi\left(p_\perp - p_\parallel\right)}{B^2} \nabla_\parallel \frac{B^2}{8\pi} = 0, \tag{65}$$

which coincides with Eq. (9.2) of Shafranov's review [29]. It is possible to prove that Eq. (65), being solvability condition to (61), reduces to an identity, by means of the statioinary kinetic equations (62) together with the quasi-neutrality condition (63). To our knowledge, first time this fact was established by J.B. Taylor [31, 32] and later by many others (see, for instance, [33–36]). Since the pressures depend on B only, Eq. (65) reduces to

$$-\frac{dp_\parallel}{dB} = \frac{\left(p_\perp - p_\parallel\right)}{B}. \tag{66}$$

In this partial case the system (64) is written as

$$\left[\mathbf{B}\times\left[\nabla\times\left(\frac{p_\perp - p_\|}{B^2}+\frac{1}{4\pi}\right)\mathbf{B}\right]\right]=0. \tag{67}$$

The existence of the identity (65) and its partial formulation (66) means that for stationary states, the pressure balance provides only two scalar equations which, together with the condition $\nabla\cdot\mathbf{B}=0$, leads to a closed system of three equations for the three magnetic field components.

It can be easilty shown that Eq. (67) can be written in the following variational form:

$$\left[\mathbf{B}\times\frac{\delta\mathcal{F}}{\delta\mathbf{A}}\right]=0 \tag{68}$$

where $\mathcal{F}$ is given by the expression

$$\mathcal{F}=\int[B^2/(8\pi)-p_\|(B)]d^3\mathbf{r},$$

and $\mathbf{A}$ is the vector potential: $\mathbf{B}=[\nabla\times\mathbf{A}]$. In the pure 2D geometry with $\mathbf{B}=(B_x,B_y,0)$ when the vector potential $\mathbf{A}$ has only one non-zeroth component ψ (z-component) for description of stationary state we arrive at the variational principle $\delta\mathcal{F}$=0, formulated in our paper [45]. It is evident also that we have the same variational principle for stationary structures in $r-z$ geometry when $\mathbf{B}=(B_r,B_z,0)$.

Note, that Eq. (64) can be written also as

$$\left[\nabla\times\left(1+4\pi\frac{p_\perp - p_\|}{B^2}\right)\mathbf{B}\right]=\chi\mathbf{B} \tag{69}$$

where for scalar function χ we have the equation $(\mathbf{B}\cdot\nabla\chi)=0$. This equation shows that χ is constant along each magnetic line. If the line is not closed so that at $r\to\infty$ $\mathbf{B}$ tends to the constant magnetic field $\mathbf{B}_0$ and besides there both pressures $p_\perp$ and $p_\|$are also constant then for all such lines $\chi=0$. Indeed, as we will show below, the equation for the stationary structures following from the guiding center formalism coincides with Eq. (69) at $\chi=0$.

C. Derivation of the varitional principle from the guiding-center formalism

To derive the variational principle for stationary mirror structures a three-dimensional, previously established for 2D configurations [45], we now employ the Hamiltonian theory of guiding-center motion as stated in Section III of Ref.[59].

Let us first consider a sort of particles with mass m and electric charge e. Instead of the particle position $\mathbf{x}$ and velocity $\mathbf{v}$, we introduce new coordinates in the phase space: position $\mathbf{X}$ of the guiding center, parallel velocity component u along the magnetic field $\mathbf{B}(\mathbf{X},t)$, magnetic moment $\mu \approx m|\mathbf{v}_\perp|^2/2B(X,t)$, and a gyroangle ζ. The dynamics of these new unknown functions is determined by the following approximate Lagrangian (see derivation in Ref.[59]), valid to the lowest order on spatial derivatives,

$$L(\mathbf{X},u,\mu,\zeta) \approx \left[\frac{e}{c}\mathbf{A}(\mathbf{X},t) + mu\mathbf{b}(\mathbf{X},t)\right]\dot{\mathbf{X}} + \frac{mc}{e}\mu\dot{\zeta} - \frac{m}{2}u^2 - \mu B(\mathbf{X},t) - e\phi(\mathbf{X},t), \tag{70}$$

where $\mathbf{A}(\mathbf{X},t)$ is the vector electromagnetic potential, $\phi(\mathbf{X},t)$ is the scalar electric potential, $\mathbf{b} = \mathbf{B}/B$ is the unit tangent vector. We see that ζ is a cyclical variable in this (adiabatic) approximation, and therefore μ is a nearly conserved quantity.

It will be important that the volume element in the non-canonical phase space $(\mathbf{X},u,\mu,\zeta)$ contains a non-constant Jacobian J,

$$d\mathcal{V} = d\mathbf{x}d\mathbf{v} = J(\mathbf{X},u)d^3\mathbf{X}dud\mu d\zeta \propto [B + \frac{mc}{e}u(\mathbf{b}\cdot \operatorname{curl}\mathbf{b})]d^3\mathbf{X}dud\mu d\zeta/(2\pi).$$

Another important formula determines velocity of guiding center in stationary fields:

$$\dot{\mathbf{X}} \approx u\mathbf{b} + \frac{mcu^2}{eB}[\operatorname{curl}\mathbf{b} - \mathbf{b}(\mathbf{b}\cdot\operatorname{curl}\mathbf{b})] - \frac{c}{eB}\operatorname{grad}(\mu B + e\phi)\times\mathbf{b}. \tag{71}$$

It follows from Lagrangian (70). This formula shows that the particle moves along the magnetic line (the first term), the second term is the drift velocity due to the centrifugal force $(\operatorname{curl}\mathbf{b} - \mathbf{b}(\mathbf{b}\cdot\operatorname{curl}\mathbf{b}) = [\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b}]$ where $(\mathbf{b}\cdot\nabla)\mathbf{b}$ is the curvature); the last term is the drift due to the mirror force and the electric force. Eq. (71) can be found in many papers, see, for example, [34].

We consider (quasi-)stationary distributions of the given sort of particles like that

$$dN = f\left(\varepsilon(\mathbf{x},\mathbf{v}),\mu(\mathbf{x},\mathbf{v})\right)d\mathcal{V} = [B + \frac{mc}{e}u(\mathbf{b}\cdot\operatorname{curl}\mathbf{b})]F'_\varepsilon(\varepsilon,\mu)d^3\mathbf{X}dud\mu d\zeta/(2\pi), \tag{72}$$

where $\varepsilon = \mu B + e\phi + mu^2/2$ is the Hamiltonian of a guiding center, and $F(\varepsilon,\mu)$ is a prescribed function of the two variables. It is assumed that $F < 0$ while $F'_\varepsilon \propto f > 0$, and $F \to 0$ as $\varepsilon \to +\infty$. It is clear that f satisfies the (collisionless) drift kinetic equation, since it depends on the exact integral of motion ε and on the approximate integral of motion μ (adiabatic invariant). Therefore there is no need in checking the hydrodynamic stationarity.

We require only two relations to close the model in a self-consistent manner: they are the Maxwell equation for a stationary magnetic field and the quasi-neutrality condition:

$$\frac{1}{4\pi}\nabla\times\mathbf{B}=\mathbf{j}_{\rm total}/c, \tag{73}$$

$$\rho_{\rm total}=0,$$

where $\mathbf{j}_{\rm total}$ and $\rho_{\rm total}$ are the densities of the electric current and of the electric charge, respectively, produced by all sorts of particles present in the system.

In the lowest order on gradients, the current density from the given sort of particles is (it follows from Eq.(3.53) of Ref.[59])

$$\mathbf{j}/c=-\nabla\times(\mathbf{b}N\langle\mu\rangle)+(e/c)N\langle\dot{\mathbf{X}}\rangle. \tag{74}$$

Here $N\langle\mu\rangle=|\mathbf{M}|=\int\mu fJdud\mu d\zeta$, where $\mathbf{M}$ is the spatial density of the magnetic moment. Using distribution (72), we have

$$-\mathbf{b}N\langle\mu\rangle=-\mathbf{B}\int\mu F'_{\varepsilon}(\varepsilon,\mu)d\mu du=-\mathbf{B}\frac{\partial}{\partial B}\int F(\varepsilon,\mu)d\mu du=\mathbf{B}\frac{\partial}{\partial B}\left(\frac{\tilde{p}_{\parallel}}{B}\right), \tag{75}$$

where $\tilde{p}_{\parallel}(B,\phi)$ is the parallel pressure of the given sort of particles,

$$\tilde{p}_{\parallel}(B,\phi)=B\int mu^2F'_{\varepsilon}(\varepsilon,\mu)d\mu du=-B\int F(\varepsilon,\mu)d\mu du.$$

It is remarkable that the calculation of $N\langle\dot{\mathbf{X}}\rangle\equiv\int\dot{\mathbf{X}}fJdud\mu d\zeta$ with the help of Eqs.(71) and (72) results in the following compact expression,

$$(e/c)N\langle\dot{\mathbf{X}}\rangle\approx\nabla\times(\mathbf{b}\tilde{p}_{\parallel}/B). \tag{76}$$

Let us now label each sort of particle by an index α. Then the Maxwell equation (73) after substitution of Eqs.(75) and (76) into Eq.(74) for each α and after subsequent summation over α looks as follows,

$$\nabla\times\left\{\mathbf{b}\left[\frac{B}{4\pi}-\frac{\partial}{\partial B}p_{\parallel}(B,\phi)\right]\right\}\approx 0, \tag{77}$$

where $p_{\parallel}(B,\phi)=\sum_\alpha\tilde{p}_\alpha$ is the total parallel pressure,

$$p_{\parallel}(B,\phi)=-\sum_\alpha B\int F_{(\alpha)}(\varepsilon_\alpha,\mu)d\mu du,$$

with

$$\varepsilon_\alpha=\mu B+e_\alpha\phi+m_\alpha u^2/2.$$

The quasi-neutrality condition

$$\sum_{\alpha} e_{\alpha} B \int \frac{\partial}{\partial \varepsilon_{\alpha}} F_{(\alpha)}(\varepsilon_{\alpha}, \mu) d\mu du = 0$$

is easily seen to have the form

$$\frac{\partial}{\partial \phi} p_{\|}(B, \phi) = 0. \tag{78}$$

Since $\operatorname{div} \mathbf{B} = 0$, equations (77) and (78) possess the variational structure, and the corresponding functional is

$$\mathcal{F} = \int [B^2/(8\pi) - p_{\|}(B, \phi)] d^3\mathbf{X}. \tag{79}$$

In principle, the quasi-neutrality condition (78) allows one to express the electric potential ϕ through B, and then the parallel pressure in Eq.(79) can be understood as a function of B only. As the result, we have the equation

$$\nabla \times \left\{ \mathbf{b} \left[\frac{B}{4\pi} - p'_{\|}(B) \right] \right\} \approx 0, \tag{80}$$

which coincides with Eq. (69) at $\chi = 0$. Thus, we get a 3D generalization of the 2D variational principle previously derived in [45] by a different approach.

It is worth noting that the quantity

$$\frac{4\pi}{c} \mathbf{j} = 4\pi [\nabla \times (\mathbf{b} p'_{\|}(B))]$$

can be connected with mean (per volume unit) magnetic moment of plasma $\mathbf{M}$=$\mathbf{b}p'_{\|}(B)$, so that the magnetic field $\mathbf{H} = \mathbf{B} + 4\pi\mathbf{M}$ (in accordance with the definition of the Maxwell equations in continuous media. In this case, equation (80) is nothing more as the Maxwell equation $\nabla \times \mathbf{H} = \mathbf{0}$.

Let us consider the most physically interesting case where the functions $F_{(\alpha)}(\varepsilon_{\alpha}, \mu)$ have the exponential on ε_{α} form,

$$F_{(\alpha)}(\varepsilon_{\alpha}, \mu) = -\exp(-\varepsilon_{\alpha}/T_{\alpha}) \tilde{D}_{\alpha}(\mu),$$

with constant temperature parameters T_{α} and some positive functions $\tilde{D}_{\alpha}(\mu)$. In this case the u-integration is simple, and

$$p_{\|}(B, \phi) = B \sum_{\alpha} T_{\alpha} \exp(-e_{\alpha}\phi/T_{\alpha}) \int_0^{+\infty} \exp(-\mu B/T_{\alpha}) D_{\alpha}(\mu) d\mu,$$

where $D_\alpha(\mu) \propto \tilde{D}_\alpha(\mu)$ by an α-dependent factor. Suppose we deal with the simplest electron-proton plasma. Then

$$p_\parallel(B,\phi) = T_i G_i(B)\exp(-e\phi/T_i) + T_e G_e(B)\exp(e\phi/T_e),$$

where

$$G_\alpha(B) = B\int_0^{+\infty}\exp(-\mu B/T_\alpha)D_\alpha(\mu)d\mu, \qquad \alpha = i, e.$$

The quasi-neutrality condition (78) now takes a simple form,

$$\exp(-e\phi/T_i)G_i(B) - \exp(e\phi/T_e)G_e(B) = 0,$$

from which we have (compare with Eq. (53))

$$e\phi = \frac{\ln[G_i(B)/G_e(B)]}{(1/T_i + 1/T_e)},$$

$$p_\parallel(B) = (T_i + T_e)[G_i(B)]^{\frac{T_i}{T_i+T_e}}[G_e(B)]^{\frac{T_e}{T_e+T_i}}. \tag{81}$$

In particular, we may assume purely thermal isotropic electron velocity distribution, which corresponds to $D_e(\mu) = const$. In that case $G_e(B) = const$, and the total parallel pressure simplifies to

$$p_\parallel(B) = n_0(T_i + T_e)\left[\frac{G_i(B)}{G_i(B_0)}\right]^{\frac{T_i}{T_i+T_e}}. \tag{82}$$

VI. TWO-DIMENSIONAL STATIONARY STRUCTURES OF THE GRAD-SHAFRANOV TYPE

In two dimensions, we define the stream function ψ (or vector potential), such that $B_x = \partial\psi/\partial y$, $B_y = -\partial\psi/\partial x$. In terms of ψ and B_z,

$$\begin{aligned}\Big[\Big[\nabla\times\mathbf{B}\Big]\times\mathbf{B}\Big] = \mathbf{e}_x\Big(-\frac{1}{2}\frac{\partial B_z^2}{\partial x} - \frac{\partial\psi}{\partial x}\Delta\psi\Big)\\ +\mathbf{e}_y\Big(-\frac{1}{2}\frac{\partial B_z^2}{\partial y} - \frac{\partial\psi}{\partial y}\Delta\psi\Big) - \mathbf{e}_z\left\{\psi, B_z\right\},\end{aligned} \tag{83}$$

where $\{\psi, B_z\}$ denotes the Jacobian. Furthermore, $\nabla_\perp = \nabla - \frac{1}{B^2}\mathbf{B}_\perp(\mathbf{B}_\perp\cdot\nabla) - \frac{B_z}{B^2}\mathbf{e}_z(\mathbf{B}_\perp\cdot\nabla)$, where $\nabla \equiv (\partial_x, \partial_y)$ and $\mathbf{B}_\perp = (B_x, B_y)$.

In Eq. (67), we now separate the (x, y)-components:

$$\begin{aligned} &-\nabla p_\perp + \frac{1}{B^2}\mathbf{B}_\perp(\mathbf{B}_\perp\cdot\nabla)p_\perp \\ &+\frac{1}{2B^2}(p_\perp - p_\|)\left[\nabla - \frac{1}{B^2}\mathbf{B}_\perp(\mathbf{B}_\perp\cdot\nabla)\right]B^2 \\ &+\frac{1}{4\pi}\left[1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right]\left(-\frac{1}{2}\nabla B_z^2 - \nabla\psi\Delta\psi\right) = 0. \end{aligned} \tag{84}$$

Due to identity (66), the equation for the z component can be written

$$\begin{aligned} &\frac{B_z}{4\pi}\left[(\mathbf{B}_\perp\cdot\nabla)\left(1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right)\right] \\ &+\frac{1}{4\pi}\left[1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right](\mathbf{B}_\perp\cdot\nabla)B_z = 0. \end{aligned} \tag{85}$$

In terms of ψ, after integration, it leads to

$$\frac{B_z}{4\pi}\left(1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right) = f(\psi). \tag{86}$$

Interestingly, in the isotropic case ($p_\perp - p_\| = 0$), we have $B_z = B_z(\psi)$, in full agreement with the Grad-Shafranov reduction [26, 27, 29]. Furthermore, because the projection of the full equation on $\mathbf{B}$ is equal to zero, in the 2D case where the fields are functions of x and y only, the projection of Eq. (84) on $\mathbf{B}_\perp$ vanishes identically. Therefore the relevant information is obtained by taking the vector product of Eq. (84) with $\mathbf{B}_\perp$, in the form

$$\begin{aligned} &\left(\nabla\psi\cdot\nabla\left[p_\perp + \frac{B_z^2}{8\pi}\right]\right) - \frac{(p_\perp - p_\|)}{2B^2}\left(\nabla\psi\cdot\nabla\left(B^2 - B_z^2\right)\right) \\ &= -\frac{(B^2 - B_z^2)}{4\pi}\left[1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right]\Delta\psi. \end{aligned} \tag{87}$$

This equation is supplemented by relation (86).

Equation (87) can be viewed as analogous to the Grad-Shafranov equation, the main difference being that the pressures are here prescribed as functions of the magnetic field amplitude. In particular, it does not reduce in the isotropic case to the usual Grad-Shafranov equation. Note that, according to the previous section, the obtained equations (86) and (87) follow from the variational principle for $\mathcal{F}$. In particular, for the purely two-dimensional geometry when $B_z = 0$ and $B^2 = |\mathbf{B}_\perp|^2$ Eq. (87) reduces to

$$\nabla\cdot\left\{\left[1+\frac{4\pi}{B^2}(p_\perp - p_\|)\right]\nabla\psi\right\} = 0 \tag{88}$$

and thus derives from the variational principle $\delta\mathcal{F} = 0$ with $\mathcal{F} = \frac{1}{4\pi}\int g(|\nabla\psi|^2)dxdy$. Here the function g is found by integrating

$$g'(B^2) = 1 + \frac{4\pi}{B^2}(p_\perp - p_\|).$$

Due to identity (66), we have

$$\mathcal{F} = \int \left(\frac{B^2}{8\pi} - p_\parallel \right) dx\, dy \equiv - \int \Pi_\parallel \, dx\, dy. \tag{89}$$

It follows that all the two-dimensional stationary states in anisotropic MHD are stationary points of the functional $\mathcal{F}$. Its density is a function of $B = |\nabla\psi|$ only. In the special case of cold electrons, this free energy turns out to identify with the Hamiltonian of the static problem [8].

Equations similar to (88) arise in the context of pattern structures in thermal convection. As shown in [30], such equations represent integrable hydrodynamic systems. As in the usual one-dimensional gas dynamics, these systems display breaking phenomena where the solution looses its smoothness at finite distance, due to the formation of folds. As a consequence, these models require some regularization. For patterns, the authors of [30] supplement in the equation an additional linear term involving a square Laplacian. In our case, this procedure corresponds to the replacement of $\mathcal{F}$ by $\mathcal{F} + (\nu/2) \int (\Delta\psi)^2 \, dxdy$, with a constant $\nu > 0$. In plasma physics, regularization can originate from finite Larmor radius (FLR) corrections, which are not retained in the present analysis based on the drift kinetic equation (see, e.g. [17, 18]). In the three-dimensional geometry the same regularization reads as (compare to [45]):

$$\tilde{\mathcal{F}} = \int \left[\frac{B^2}{8\pi} - p_\parallel(B) + \frac{\nu}{2} |\nabla \times \mathbf{B}|^2 \right] d^3\mathbf{r}. \tag{90}$$

One more remark. Let $\mathbf{B}$ be a function of x and y, but $B_z \neq 0$. In this case B_z is not defined by stream function ψ and therefore one needs to write down

$$\left[\mathbf{B} \times \left[\nabla \times \frac{\delta \mathcal{F}}{\delta \mathbf{B}} \right] \right] = 0. \tag{91}$$

Hence it is easily to get Eqs. (9,10) from [45]. It is necessary to mention also that for the 2D case the stationary states with $B_z \neq 0$ are determined from the equation

$$\left[\nabla \times \frac{\delta \mathcal{F}}{\delta \mathbf{B}} \right] = 0.$$

For instance, the equation for B_z has the form

$$\left(1 + 4\pi \frac{p_\perp - p_\parallel}{B^2} \right) B_z = \text{const},$$

where instead of arbitrary function of ψ (see Eq. (9) from [45]) we have const. In $r - z$ geometry the analogous situation takes place where B_φ plays the same role as B_z in the planar case.

Note that for isotropic plasma ($p_\perp = p_\|$), H. Grad and H. Rubin [28] formulated for the stationary MHD states the variational principle for

$$\mathcal{F} = \int \left(\frac{B^2}{8\pi} - p \right) d\mathbf{r}.$$

A. KP soliton

We shall now show that the functional $\mathcal{F}$ we previously introduced has the meaning of a free energy. In the weakly nonlinear regime near the MI threshold, the temporal behavior of the mirror modes can be described by a 3D model [17, 18, 24], that in the present 2D geometry reads

$$u_t = -\widehat{|k_y|} \frac{\delta F}{\delta u} \tag{92}$$

with the free energy

$$F = \int \left[\frac{1}{2}(-\varepsilon u^2 + u \frac{\partial_z^2}{\Delta_\perp} u + (\nabla_\perp u)^2) + \frac{\lambda}{3} u^3 \right] d\mathbf{r}. \tag{93}$$

Here u denotes the dimensionless magnetic field fluctuations and ε the distance from MI threshold. The third term in F originates from the FLR corrections, and λ is a nonlinear coupling coefficient which is positive for bi-Maxwellian distributions. In Eq. (92), the operator $\widehat{|k_y|}$ is a positive definite operator (in the Fourier representation it reduces to $|k_y|$), so that Eq. (92) has a generalized gradient form.

Let us now show that this result can be obtained from the functional $\mathcal{F}$ defined in (89). We isolate the perturbation φ in the stream function $\psi = -B_0(x + \varphi)$ with $\varphi \to 0$ as $|\mathbf{r}| \to \infty$, so that the mean magnetic field $\mathbf{B}_0$ is directed along the y-axis. We then expand Eq. (89) in series with respect to u. For the sake of simplicity, we restrict the analysis to the case of cold electrons. The expansion of the integrand $B^2/(8\pi) - p_\|$ in F has then the form

$$\begin{aligned} n_0 T_\| & \left[\frac{(u+1)^2}{\beta_\|} - \frac{1+u}{1+au} \right] \\ & = n_0 T_\| [\left(\beta_\|^{-1} - 1 \right) + u \left(a + 2\beta_\|^{-1} - 1 \right) \\ & +u^2 \left(-a^2 + a + \beta_\|^{-1} \right) - u^3 a^2 (a-1) + ...] \end{aligned} \tag{94}$$

where we use the usual notation $\beta_\| = 8\pi n_0 T_\| / B_0^2$.

As well known (see, e.g. [17, 18]), near threshold, MI develops in quasi-transverse directions relative to $\mathbf{B}_0$. This means that, in the 2D geometry, $\varphi_x \gg \varphi_y$ and, with a good accuracy, u coincides with φ_x. However, in the expansion of $u = \sqrt{(\varphi_x+1)^2+\varphi_y^2} - 1 \simeq \varphi_x + \varphi_y^2/2$, it is necessary to keep the second term, quadratic with respect to φ. The linear term in the expansion of $\mathcal{F}$ vanishes and the quadratic terms is given by

$$\mathcal{F}_2 = n_0 T_\parallel \int \Big\{ \Big[a(a-1) + \frac{1}{\beta_\parallel}\Big]\varphi_x^2 + \Big[a - 1 + \frac{2}{\beta_\parallel}\Big]\varphi_y^2 \Big\} dxdy.$$

where the factor $a(a-1)+1/\beta \equiv -\varepsilon/2$ defines the MI threshold $a = 1+1/\beta_\perp$ (that the present equations of state accurately reproduces). It is also seen that for $|\varepsilon| \ll 1$, $\varphi_x/\varphi_y \sim |\varepsilon|^{-1/2}$, in agreement with the quasi-one-dimensional development of MI near threshold. In this case, $\mathcal{F}_2$ coincides with the quadratic term in (93), up to a simple rescaling and to the FLR contribution, Furthermore, the cubic term in (94) gives the nonlinear coupling coefficient $\lambda = a\,(a-1) > 0$. As a consequence, $\mathcal{F}$, introduced in the previous section, reduces to the free energy of the asymptotic model. The temporal equation for φ has also the generalized gradient form originating from (92),

$$\varphi_t = -\Gamma \frac{\delta F}{\delta \varphi} \text{ with } \Gamma = -\frac{\widehat{|k_y|}}{k_x^2}, \tag{95}$$

for which the associated stationary equation reads

$$\varepsilon \varphi_{xx} + \varphi_{xxxx} - \varphi_{yy} - \lambda \partial_x \left(\varphi_x^2\right) = 0, \tag{96}$$

where the linear operator $L = -\varepsilon \partial_{xx} + \partial_{yy} - \partial_{xxxx}$ is elliptic or hyperbolic depending on the sign of ε. For $\varepsilon > 0$ (above threshold), this operator is hyperbolic, while below threshold it is elliptic and thus invertible in the class of functions vanishing at infinity. Remarkably, in the latter case, Eq. (96) identifies with the soliton for KP equation called lump. In standard notations, lump is indeed a solution of the stationary KP-II equation,

$$-Vu_{xx} + u_{xxxx} - u_{yy} + 3(u^2)_{xx} = 0, \tag{97}$$

where V is the lump velocity. When comparing this equation with (96) we see that $-|\varepsilon|$ plays the role of the lump velocity V and $\lambda\varphi_x \to -3u$.

The lump solution was first discovered numerically by Petviashvili [57] using the method now known as the Petviashvili scheme (see the next section). The analytical solution was

later on obtained in [58]. In our notation, it reads

$$\varphi_x = -\frac{12|\varepsilon|}{\lambda}\frac{(3+\varepsilon^2 y^2 - |\varepsilon|x^2)}{[3+\varepsilon^2 y^2 + |\varepsilon|x^2]^2}.$$

This function vanishes algebraically at infinity like r^{-2}. In the center region $-|\varepsilon|^{-2}\sqrt{|\varepsilon|x^2-3} < y < |\varepsilon|^{-2}\sqrt{|\varepsilon|x^2-3}$, the magnetic field displays a hole with a minimum at $x = y = 0$ equal to $-4|\varepsilon|/\lambda$. In the outer region, the magnetic lump has two symmetric humps with maximum values $|\varepsilon|/(2\lambda)$ at $y = 0$ and $x = \pm 3|\varepsilon|^{-1/2}$. The main contribution to the "skewness" $I = \int \varphi_x^3 \, dx \, dy$ comes from the hole region, providing a negative value to I, in complete agreement with [17, 18].

VII. NUMERICAL 2D SOLUTIONS

In the 2D case, our regularized model equation for stationary pressure-balanced structures has a variational form

$$-\partial_x\left[\frac{(1+\varphi_x)}{(1+u)}\frac{dg}{du}\right] - \partial_y\left[\frac{\varphi_y}{(1+u)}\frac{dg}{du}\right] + \nu\Delta^2\varphi = 0. \tag{98}$$

Clearly, Eq. (98) describes stationary points $\delta\mathcal{F}/\delta\varphi = 0$ of the functional $\mathcal{F} = \int [g(u) + (\nu/2)(\Delta\varphi)^2]\, dx\, dy$, with some constant parameter ν (in this expression and everywhere below, we use dimensionless variables).

We applied two numerical methods to solve Eq. (98). The first one is a generalization of the well known gradient method which corresponds to a dissipative dynamics along an auxiliary time-like variable τ of the form $\varphi_\tau = -\widehat{\Gamma}(\delta\mathcal{F}/\delta\varphi)$, with a positive definite linear operator $\widehat{\Gamma}$. It is clear that attractors in the phase space of the above dynamical system are stable solutions of Eq. (98). Unstable solutions however cannot be found by this method.

Furthermore, the linear part of Eq. (98) is of the form $\widehat{L}\varphi = -g''(0)\varphi_{xx} - g'(0)\varphi_{yy} + \nu\Delta^2\varphi$. The coefficient $g''(0)$ is proportional to ε (introduced in the previous section) and $g'(0)$ is positive within the adiabatic approximation. When these two are positive, the operator L is elliptic and it is possible to employ the so-called Petviashvili method [57]. It is a specific method for finding localized solutions of equations of the form $\widehat{M}\varphi = N[\varphi]$, with a positively definite linear operator $\widehat{M}$ and a nonlinear part $N[\varphi]$. Note that in our case the Fourier image of $\widehat{M}$ is

$$M(k_x, k_y) = g''(0)k_x^2 + g'(0)k_y^2 + \nu(k_x^2 + k_y^2)^2 > 0. \tag{99}$$

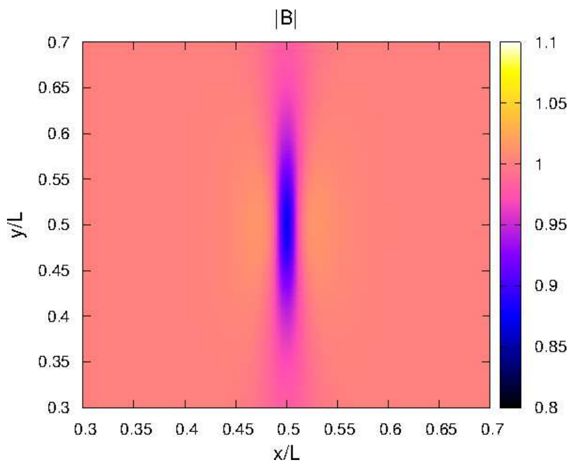

FIG. 3: Fig. 1. Unstable localized solution for $\nu = 0.0004$, $r = 7$, $B_s = 0.5$ (in units of B_0), and $C = 0.002$. The value $1/\beta_{\|} = 1.127$ prescribes an aspect ratio $\sqrt{g''(0)/g'(0)} = 0.2$.

In its simplest form, the iteration scheme of the Petviashvili method reads

$$\varphi_{n+1} = (\widehat{M}^{-1} N[\varphi_n]) \left(\frac{\int \varphi_n \widehat{M} \varphi_n \, dx \, dy}{\int \varphi_n N[\varphi_n] \, dx \, dy} \right)^{-\gamma}, \tag{100}$$

where γ is a positive parameter in the range $1 < \gamma < 2$. The corresponding multiplier strongly affects the structure of attractive regions in the phase space.

It is worth noting that if the operator $\widehat{L}$ is hyperbolic, solutions of the problem are not localized with respect to both x and y coordinates, and will be periodic or more generally quasiperiodic [48, 50].

A. The results

We performed computations with both numerical methods using fast Fourier transform numerical routines for the evaluation of the linear operators. Periodic boundary conditions for a computational square $2\pi \times 2\pi$ were assumed.

For the gradient method, we used the simplest first-order Euler scheme for stepping along τ, with $\delta\tau \sim 0.01$. The operator $\widehat{\Gamma}$ was taken in a form giving stable computation, namely $\Gamma(k_x, k_y) = 1/[k_x^2 + k_y^2 + \nu(k_x^2 + k_y^2)^2]$.

As for the Petviashvili method, the value $\gamma = 1.8$ was used, leading, after an erratic transient, to a convergence of the iterations to unstable solutions of the variational equation (98).

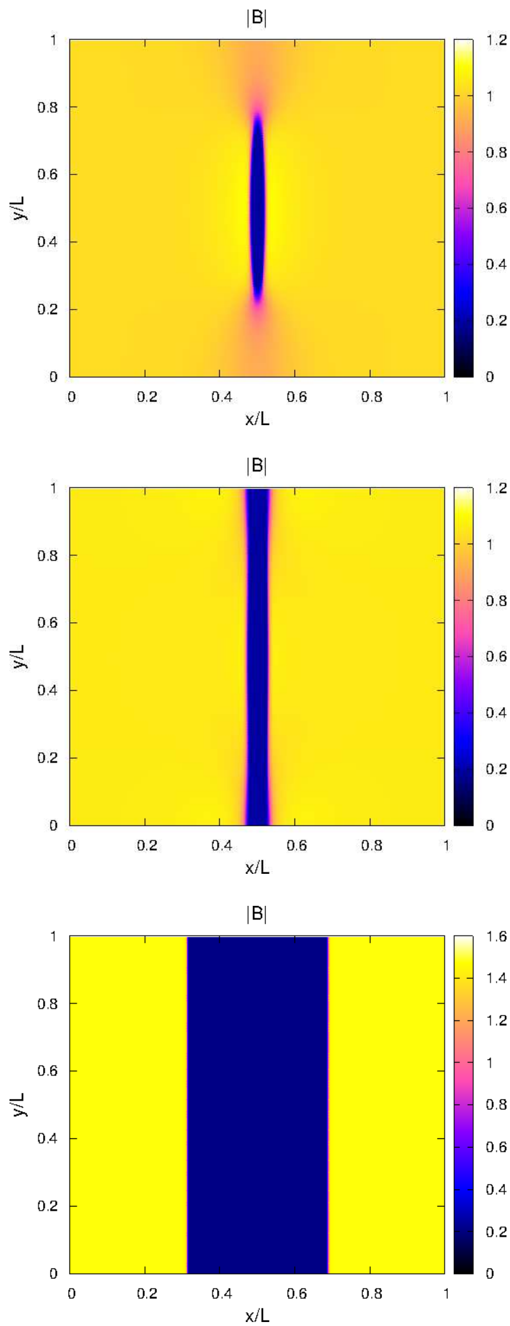

FIG. 4: Fig. 2. Formation of a stable 1D solution in a gradient computation, for the same parameters as in Fig.1.

The main results of our computations can be formulated as follows. There do exist unstable localized solutions of Eq. (98), which are similar to the lump solutions of KPII equation, when written in terms of $u = \partial_x \varphi$ (Fig. 3). For asymptotically small ε, they accurately coincide with KP solutions, independently of the electron temperature, as it

should be. Such low-amplitude stationary states do not depend on the particular choice of the regularization of $g(u)$. No other kinds of solutions were found with the Petviashvili method.

When the gradient method is used, large amplitudes $u \sim 1$ are achieved in many cases, and the final result turns out to be dependent on the choice of the parameters r and C in the regularized function g. Without regularization, no smooth stationary state is approached. Instead, a singularity occurs. Differently, when a regularized g with parameters $r \sim 10$ and $C \sim 0.001$ is used, the final state identifies with a one-dimensional stripe in the form of a magnetic hole, as shown in Fig. 4 that also displays typical stages of the "gradient" evolution. In all simulations, the magnetic field in the stripe was smaller than the 'singular' magnetic field B_s given by Eq. (56). For increasing r, the magnetic field in the stripe tends to decrease, down to 0. For initial conditions in the form of a slightly perturbed 2D lump, the final result is always a one-dimensional stripe of hole type, which demonstrates the instability of the 2D lump, in full agreement with the analytical prediction [17, 18].

In no cases stable 2D structures localized both in x and y directions were found. Instead, the gradient method showed that stable structures can only be one-dimensional, transverse to the magnetic field. An initial localized perturbation of sufficiently high amplitude develops into an increasingly long structure along the y axis, and eventually reaches the boundary of the computational domain.

The question arises whether the 1D shock solutions obtained in [8] (for which $\min B > B_s$) would identify with the present solution when $\nu \to 0$, a limit which is unreachable in the present numerics. It is possible that the presence of the bi-Laplacian regularization leads to overshooting in the shock solution, resulting in the convergence towards solutions where $\min B < B_s$.

B. 2D mirror structures with $B_z \neq 0$: stripes and magnetic bubbles

Let us consider some numerical examples. For simplicity we take the function $D_i(\mu) \propto \exp(\mu B_s/T_i)$ for $\mu < \mu_*$, and $D_i(\mu) = const$ for $\mu > \mu_*$, with some parameter $B_s < B_0$, and a large μ_*. Such constant-like behaviour of $D_i(\mu)$ at very large μ is necessary both from formal and physical points of view (see discussion in Ref.[45]). At $B = B_0$ we thus have a nearly Gaussian ion perpendicular velocity distribution with the temperature $T_\perp(B_0) =$

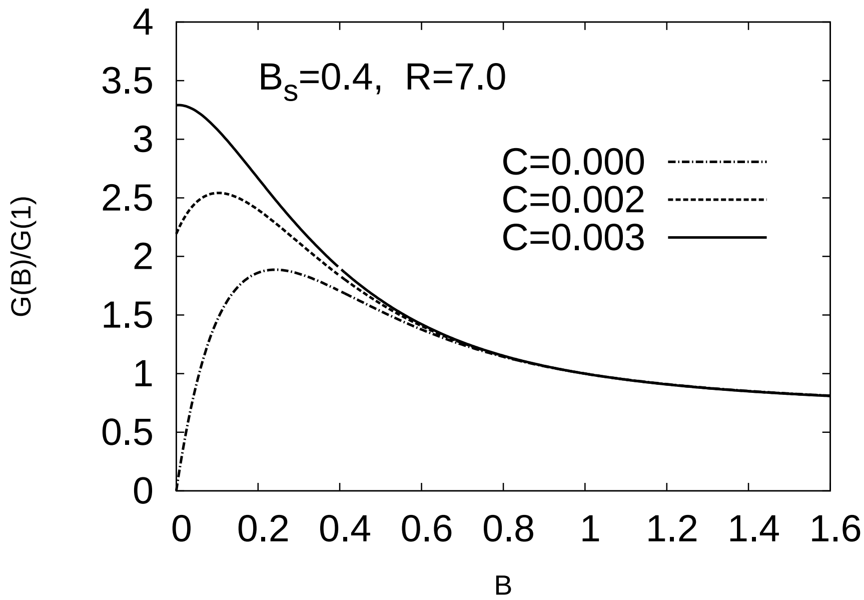

FIG. 5: Some plots corresponding to expression (101).

$T_i/(1 - B_s/B_0) > T_i$. The distribution becomes strongly non-Gaussian as the magnetic field decreases to values $B \lesssim B_s$. Let us normalize all magnetic field values to B_0 so that formally $B_0 = 1$. As the result, we have the following expression for the ratio $G_i(B)/G_i(1) \equiv W(B)$,

$$W(B) = \frac{B(1-C)(1-B_s)}{\{1 - \exp[-R(1-B_s)]\}} \frac{\{1 - \exp[-R(B-B_s)]\}}{(B - B_s)} + C \exp[R(1-B)], \qquad (101)$$

with a sufficiently large regularizing parameter R and a small parameter C. Some plots, with $B_s = 0.4$, $R = 7.0$, for several C, are shown in Fig.5

We substituted this dependence into Eq.(82) and then into Eq.(90), with $T_e = 0$. To find stable stationary 2D mirror structures with $B_z \neq 0$, we parametrized magnetic field in the following manner,

$$B_x = -\psi_y(x, y), \qquad B_y = \psi_x(x, y), \qquad B_z = \gamma(x, y).$$

We fixed mean values $\langle B_x \rangle = 0$, $\langle B_y \rangle = \cos\Theta_0$, $\langle B_z \rangle = \sin\Theta_0$. Then we employed the gradient numerical method described in Ref.[45] [with a simple generalization to include $\gamma(x, y)$] to find minimum of the functional

$$\tilde{\mathcal{F}}_{2D} = \int \Big[g\Big(\sqrt{|\nabla\psi|^2 + \gamma^2}\Big) + \frac{\nu}{2}(|\Delta\psi|^2 + |\nabla\gamma|^2) \Big] d^2\mathbf{X}, \qquad (102)$$

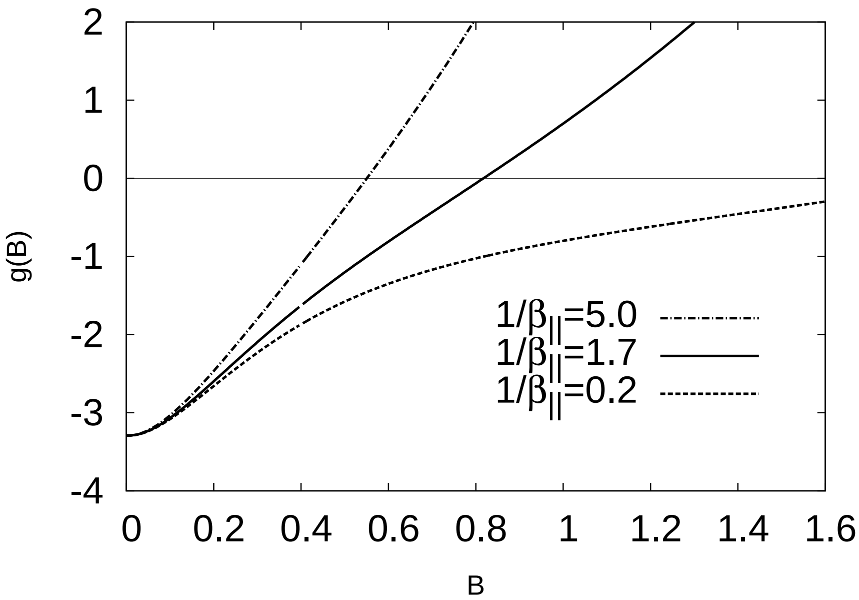

FIG. 6: Some examples corresponding to expression (103). Here $B_s = 0.4$, $R = 7.0$, $C = 0.003$. System with $1/\beta_{||} = 0.2$ is linearly unstable. System with $1/\beta_{||} = 1.7$ is linearly stable, but subcritical structures are possible. System with $1/\beta_{||} = 5.0$ is stable, no structures are possible.

where

$$g(B) = \frac{B^2}{\beta_{||}} - W(B), \tag{103}$$

and $\beta_{||} = 8\pi n_0 T_i/B_0^2$. Plots of function $g(B)$, for several values of $\beta_{||}$, are shown in Fig.6. The mirror instability takes place when the second derivative $g''(1)$ is negative. Subcritical mirror structures are possible when $g''(1)$ is positive, but there is a range of B where $g''(B) < 0$.

It is important that besides purely 1D stable configuration ("stripes"), in our computations we have detected for some parameters also essentially 2D stable solutions — "bubbles", as shown in Fig.7 for $B_s = 0.4$, $R = 7.0$, $C = 0.003$, $\langle B_y \rangle / B_0 = 0.2$, $1/\beta_{||} = 1.71$. In general, "bubbles" takes place when B_z dominates, i.e. $\cos \Theta_0$ is sufficiently small. They have the perfect circular shape in the case when $B_x = 0$ and $B_y = 0$ (see Fig.8). In all cases we have inequalities $g''(B_{\text{in}}) > 0$, and $g''(B_{\text{out}}) > 0$, so the unstable range of B is passed in the vicinity of the bubble boundary. When $B_\perp = 0$ the magnetic fields are constant inside and outside circle everywhere accept transient layer which is defined by the FLR. The size of the circular patch is defined by two factors: the conservation of magnetic field flux and the cell size. The FLR introduces small input in the this constraint, it plays a role of the surface

tension.

In Fig 9 is shown for circular bubbles the diagram of all possible both stable and unstable states at the fixed $\beta_{\|}$ measured by the B_s field. Because of the magnetic fields outside and inside the bubbles are constant, stability and instability of each state is defined by the second derivative of the function $g(B)$. At the given $\beta_{\|}$ the B_1 and B_2 curves represent the inner and outer magnetic fields when FLR is absent. The FLR in this case provides a transient solution matching the inner and outer regions. But to say that these are the inner or outer solution one needs to have another jump, or some patch if we speak about two-dimensional structures. Both states B_1 and B_2 are linearly stable. These states satisfy the necessary boundary conditions, namely, continuity of the magnetic field: $H_1 = H_2 \equiv H$, where H is an additional constant. These states, thus, can be considered as conjugated states, or, by another words, these are bistable states. When changing $\beta_{\|}$ which is defined has a meaning of the parameter ε we move along the curves B_1 and B_2. One should mention that in this case $\beta_{\|}$ is some auxiliary dimensionless parameter. Real $\beta_{\|}$ is found depending on a state by means of B_1 or B_2. If one considers any state B, say, at the given $\beta_{\|}$, without any conjugation, then one can get linear stability or linear instability by analyzing the second derivative sign of the function $g(B)$. The second point is that by fixing two conjugated states one can say only that $B_2 > B_1$. Only in the case when you have another jump one can say whether it is a hole or a hump. One more point is that the case considered here corresponds to the pure B_z case when $B_\perp = 0$.

VIII. CONCLUSION

In the first part of this paper we presented a review of our results concerning the weakly nonlinear regime of the mirror instability in the framework of the so-called asymptotic model. This model was demonstrated to belong to the class of the gradient systems for which the free energy can decrease in time only. In particular, it was shown that the stationary localized solutions of the model, below the mirror instability, occur unstable and, above the threshold, the system has a blow-up behavior up to amplitude comparable with a mean magnetic field that is typical for subcritical bifurcation. We showed also that account of electrons (increase their temperature) does not change the structure of the asymptotic model. For bi-Maxwellian distribution functions for both electrons and ions all analyzed

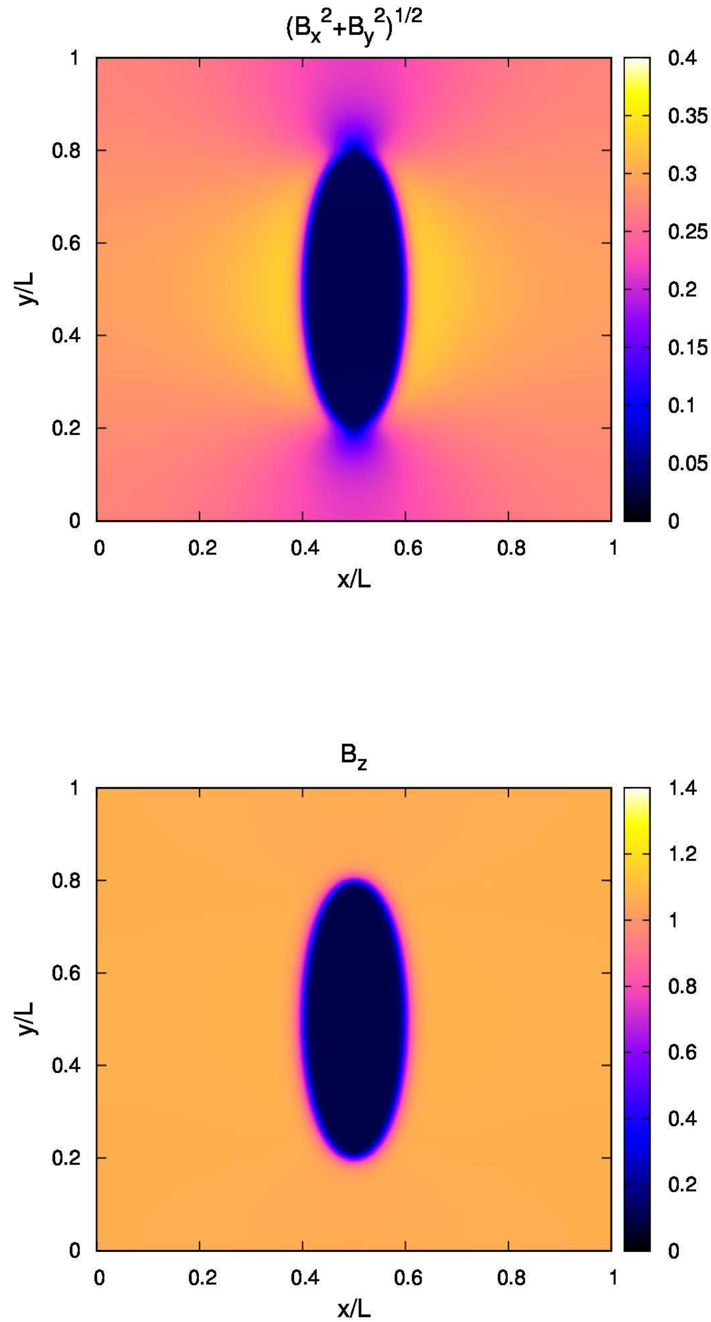

FIG. 7: Example of 2D "bubble", with $1/\beta_{\|} = 1.71$, $\langle B_y \rangle / B_0 \equiv \cos\Theta_0 = 0.2$.

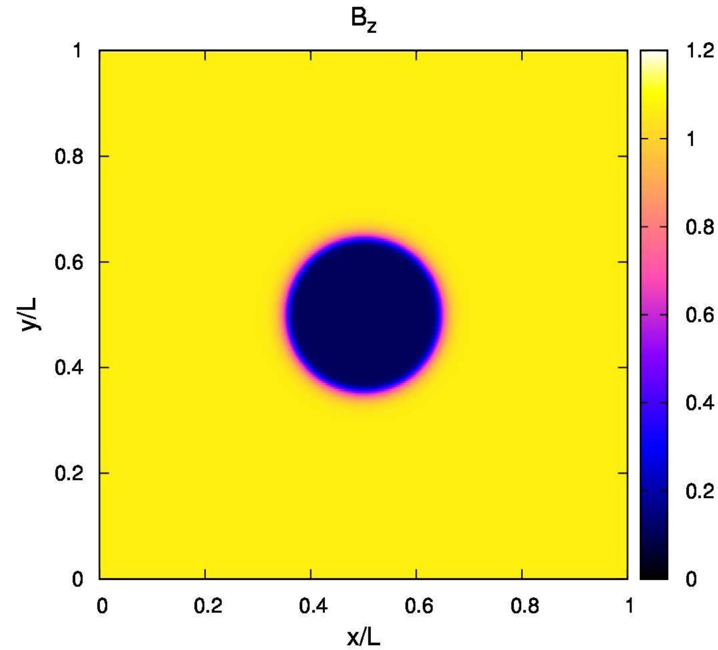

FIG. 8: Circular "bubble", with $1/\beta_{\|} = 1.71$, $\cos\Theta_0 = 0.0$.

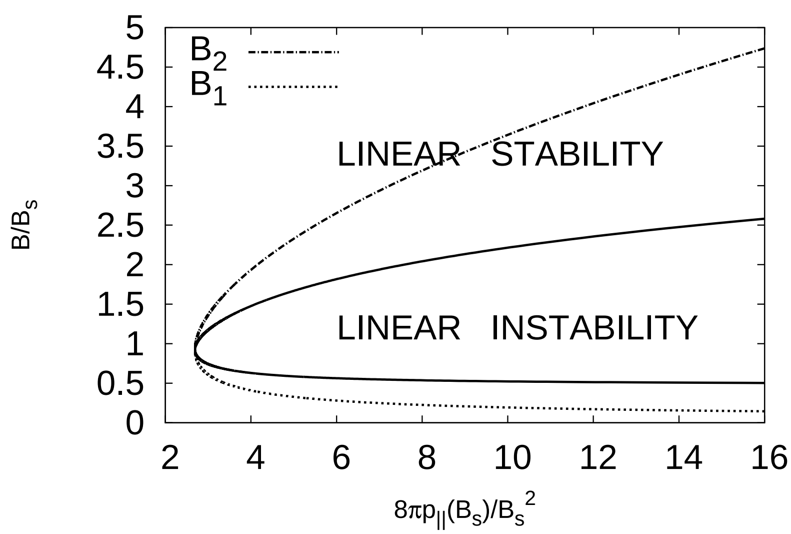

FIG. 9: B_1-B_2 diagram of stationary states depending on $\beta_{\|}$ measured by B_s

structures within the model have the form of magnetic holes. Humps can appear for distributions different from the bi-Maxwellian ones. For instance, such situation is possible after a stage of quasi-linear relaxation (for details see results of numerics [37]). The second part of this paper contains original results concerning the possible two-dimensional mirror structures which can be formed at the saturation regime of subcritical bifurcation. In particular, a detailed analysis was presented for the Grad-Shafranov equations describing static force-balanced mirror structures with anisotropic pressures given by equations of state derived from drift kinetic equations, when assuming an adiabatic evolution from bi-Maxwellian initial conditions. It turns out that in two dimensions, the problem is amenable to a variational formulation with a free energy provided by the space integral of the parallel tension. Slightly below the mirror instability threshold, small amplitude solutions associated to KPII lumps are obtained and shown to be unstable. Based on the variational computation (the gradient method) of the stationary mirror structures, this instability is shown to result in appearance of one-dimensional stripes when the magnetic fields outside and inside stripes are homogeneous with a jump which structure is defined by the FLR effects. Such two-dimensional evolution of the stationary structures are formed for below and above threshold of the mirror instability when the B_z-component of magnetic field is absent. For the finite but small enough values of B_z the resulting structures represent stripes. With increasing B_z instead of stripes we observed in numerical simulations the formation of magnetic bubbles with the homogeneous magnetic field inside the bubbles. When B_z becomes larger $B_\perp$ the form of bubbles change their form from elliptic to the circular one when $B_\perp = 0$. In the latter case, the magnetic field outside and inside bubbles occurs constant and undergoes jump due to the FLR effects while crossing the bubble. In this case, the FLR effects play the role of surface tension. Note also, when considering stable subcritical structures, the drift kinetic approximation breaks down, as the deep magnetic holes obtained by a gradient method appear to be strongly sensitive to the regularization process, an effect which in a more realistic description could be provided by FLR corrections and/or particle trapping.

Acknowledgments. This work was supported by CNRS PICS programme 6073 and RFBR grant 12-02-91062-CNRS-a. The work of E.K. and V.R. was also supported by the RAS Presidium Program " Nonlinear dynamics in mathematical and physical sciences" and Grant

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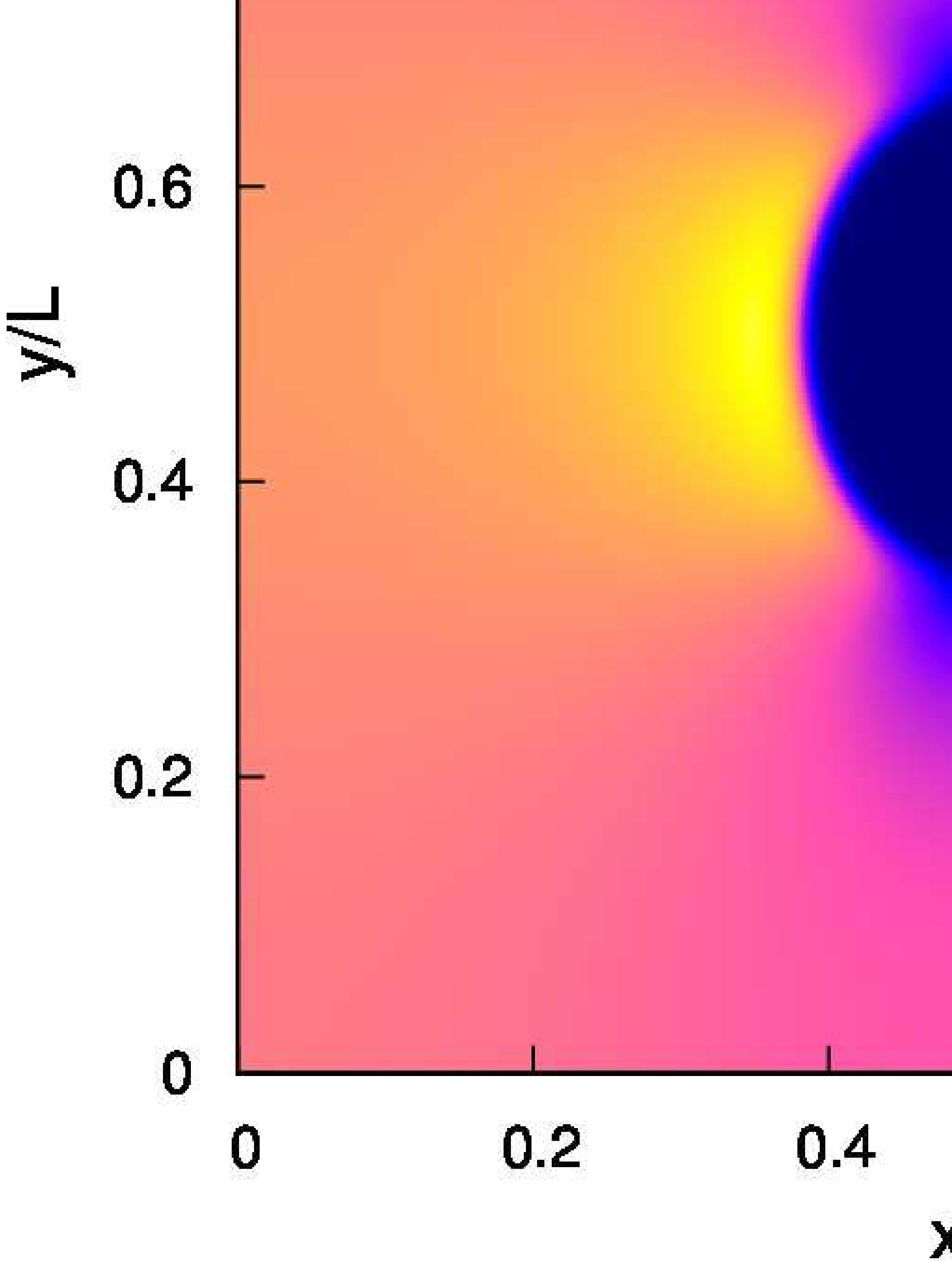

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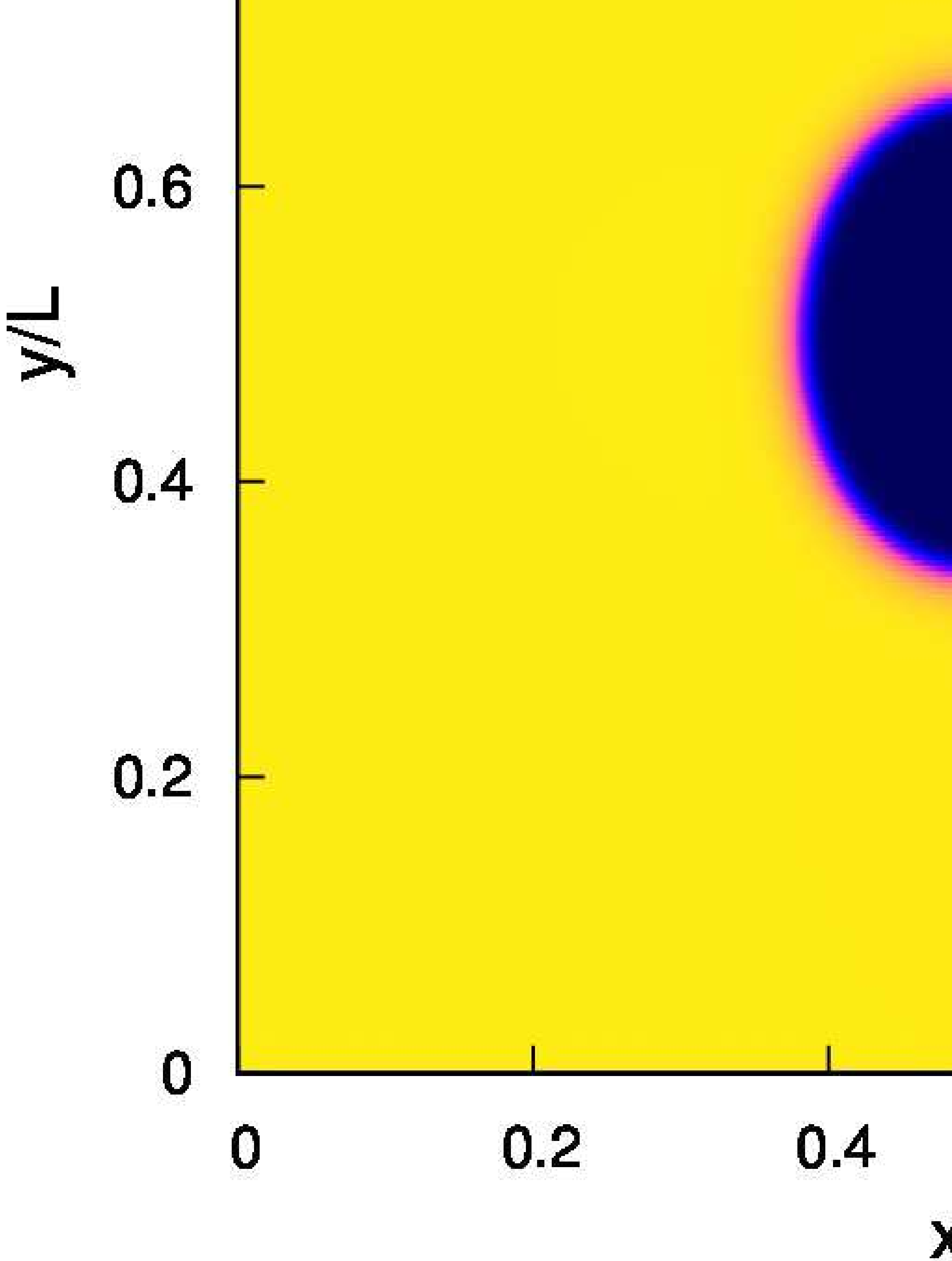

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